Stable Homotopy Theory

① Stable Homotopy Groups

Freudenthal Suspension Theorem

If X is a (CGWH) space with non-degenerated basepoint,

then $\cdots \pi_{i+n}(\Sigma^i X) \xrightarrow{\sim} \pi_{i+n+1}(\Sigma^{i+1} X) \cdots$ is iso for $i >> n$

$\pi_*^s(X) = \varinjlim_i \pi_{*+i}(\Sigma^i X)$, $\pi_*^s(-)$ is a homology theory

but $\pi_*(-)$ is not a homology theory

② Generalized (co)Homology Theory (in Top_*)

En : $Top_*[W^{-1}] \simeq h(CW_*) \longrightarrow Ab$

Satisfying stable axiom

1. A natural iso $h_n(X) \to h_{n+1}(\Sigma X)$
2. $h_n(A) \to h_n(X) \to h_n(Cf)$ is exact
3. $\oplus_\alpha h_n(X_\alpha) \xrightarrow{\simeq} h_n(\vee_\alpha X_\alpha)$

$$\begin{tikzcd} & Ho(spectra) \arrow[d, "\text{Brown rep}"] \\ Ho(Top_*) \arrow[ur, "\Sigma^\infty", "\text{stablization}"'] \arrow[r] & D \end{tikzcd}$$

stable axiom

just idea
not strict

So we only need consider (co)homology theory in Ho(spectra)

Stable Homotopy ≃ Spectra

Classical spectra

Only Ho(Spectra)

(Adams, Switzer)

Ho(S) is Triangulated Cat* by

Cofibre sequence $X \to Y \to C_f \to \Sigma X$

$\Sigma X = X[1]$ is auto equivalence

Disadvantage 1. All things are homotopy

2. Smash product is extremely ugly, ~~unnature~~, technical.

3. Homotopy module spectra Cat is not stable (triangulated)

[Equivariant stable homotopy theory (LMS) 1986

Modern spectra

- S-module (EKMM cell theorem)
- orthogonal spectra
- symmetric spectra (Schwede)
- ∞-Cat's approach

CGWH

Topological → $Ho(A,B) = \pi_0 Map(A,B)$

Monoidal

Stable

Model Category

Monoidal: ① pushout-product Axiom ② Unit Axiom ⇒ Ho(C) is monoidal

Stable: pointed model cat C, Σ: Ho(C) is auto equivalence ⇒ Ho(C) is triangulated

※ Advantage 1. Can do something strict

2. Through monoidal model structure. We have both strict smash product* Sp and homotopy smash over Ho(Sp).

Therefore, we can define E_∞-ring spectra, A_∞-spectra as commutative monoid and monoid object ~~over~~ from strict smash product

3. strict module spectrum Cat is stable

(Very useful, e.g. MU-Mod)

$$\begin{tikzcd} & MU\text{-}Mod \\ E_\infty \text{ space} / BO \arrow[r, shift left, "\text{Thom spectrum functor}"] & E_\infty\text{-ring spectra} \arrow[l, shift left] \arrow[u, "E_\infty"'] \end{tikzcd}$$

from

How to verity a model Category?

C is a category, I is a class of morphisms in C.

Def I-cof is a class of morphisms, s.t. I-fib

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "\in I"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

$f \in I\text{-cof} \Leftrightarrow \forall$ square, $\exists$ ↗

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "I \ni"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

$f \in I\text{-fib} \Leftrightarrow \forall$ square, $\exists$ ↗

0. $I \subset (I\text{-cof})\text{-fib}$, $I \subset (I\text{-fib})\text{-cof}$

Prop. 1. $I \subset J \Rightarrow I\text{-cof} \supset J\text{-cof}$ and $I\text{-fib} \supset J\text{-fib}$.

2. I-cof and I-fib are stable under retract

3. I-cof is stable under pushout. I-fib is stable under pull-back

pushout

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f_1"'] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r] & \cdot \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$f_1 \in I\text{-cof}$ $\Downarrow$ $f_2 \in I\text{-cof}$

pull-back

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f_1"'] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

$f_2 \in I\text{-fib} \Rightarrow f_1 \in I\text{-fib}$

Recall: Def of model cat

(C, cof, fib, W)

⓪ Bicomplete ① 2-out-3 for W

② cof, Fib, W are stable under retract

③
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\text{cof} + W"'] & \cdot \arrow[d, "\text{Fib}"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}
\qquad
\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\text{Cof}"'] & \cdot \arrow[d, "\text{Fib} + W"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

√ by Def

④
$$\begin{tikzcd} & X' \arrow[dr, "\text{Fib}"] & \\ X \arrow[ur, "\text{cof}+W"] \arrow[rr, dashed] \arrow[dr, "\text{Cof}"'] & & Y \\ & Y' \arrow[ur, "\text{Fib}+W"'] & \end{tikzcd}$$

What we do usually?

1. Given a set of morphisms I, defining Fib = I-fib,
"+" = 且 = ∩
Cof = (Fib ∩ W)-cof. (In Top, $I = \left\{ \begin{smallmatrix} D^n \times 0 \\ \downarrow \\ D^n \times I \end{smallmatrix} \right\}$)

need

2. prove $\varinjlim_n C(F, X_n) \xrightarrow{\cong} C(F, \varinjlim_n X_n)$ $\forall F$ ~~a~~ the domain of a morphism $f \in I$

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \cdots & X_n \arrow[r] & X_{n+1} \cdots \\ & & \coprod F_\alpha \arrow[u] \arrow[r] & \coprod_\alpha E_\alpha \arrow[u] \arrow[ul, phantom, "\text{push}"] \end{tikzcd}$$

$\begin{matrix} F_\alpha \\ \downarrow \\ E_\alpha \end{matrix} \in I$

3. By small object argument, $\forall X \to Y$, we get a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend right] & \cdot \arrow[r, "p"] & Y \end{tikzcd}$$

, $p \in I\text{-fib}$, $i \in (I\text{-fib})\text{-cof}$
$\cap$
$\text{Cof} = (I\text{-fib} \cap W)\text{-cof}$

Need prove $i \in W$

4. Need prove $\text{Fib} \cap W = J\text{-fib}$ for ~~given~~ some set of morphisms (In Top, $J = \left\{ \begin{smallmatrix} S^{n-1} \\ \downarrow \\ D^n \end{smallmatrix} \right\}$)

5. By trand argument, we get Axiom ③

$$\begin{tikzcd} F \arrow[d, equal] & E \arrow[l, "f"'] \arrow[r] \arrow[d, "i"] & \cdot \arrow[d, "\text{Fib}"] \\ F & E' \arrow[l, "p"] \arrow[r] & \cdot \end{tikzcd}$$

6. similarly as step 3, we get Axiom ④

Nontrivial extra Inputs:

1. step 2.

2. i in step 3 is in W

3. step 5.

① CGWH

② Category

EKMM's framework

4.

$$\underset{\text{}}{P\mathcal{U}} \rightleftarrows \overset{\text{Ho}(S\mathcal{U})\text{ model}}{S\mathcal{U}} \rightleftarrows \overset{\text{weakly monoidal model cat}}{\mathbb{L}\text{-Spectra}} \rightleftarrows \overset{\text{monoidal model cat}}{S\text{-module}}$$

$(\mathcal{U} \simeq \mathbb{R}^\infty)$

Prespectra:

~~$(\mathcal{U}, A)$~~ index set $(\mathcal{U}, A)$, where $\mathcal{U}$ is an real inner space of countable infinite dimension

A is a cofinal subset of $\left\{ \begin{array}{l} \text{finite dimension} \\ \text{subspace of } \mathcal{U} \end{array} \right\}$

continuous functor

$A \to Top_*$ $\quad V \subset W$

$\Sigma^{W-V} E_V \underset{\sigma}{\to} E_W$ — associated — unital

$E_V \xrightarrow{\tilde{\sigma}} \Omega^{W-V} E_W$ is homeomorphism $\Leftarrow\Rightarrow$ E is a spectrum

difficulty 1

$$\begin{tikzcd} PA \arrow[r, shift left, "L"] & SA \arrow[l, shift left, "l"] \\ PB \arrow[u] \arrow[r, shift left, "L"] & SB \arrow[l, shift left, "l"] \arrow[u, "\simeq"'] \end{tikzcd}$$

The construction L is technical

(Subtle fact in CGWH: Given a sequence of (not necessarily closed) inclusions in CGWH $X_1 \to X_2 \cdots X_n$,

then $\varinjlim_n \Omega X_n \underset{同胚}{\overset{\simeq}{\to}} \Omega(\varinjlim X_n)$

difficulty 2

Twisted half-smash product

$$S\mathcal{U}'(A \ltimes E, E') \simeq \text{~~}Top \downarrow L(\mathcal{U},\mathcal{U}')(A, S(E, E')) \simeq S\mathcal{U}(E, F[A, E'))$$

~~$\Sigma^\infty_V S^0 \simeq S^{-V}$~~

$$\Pi^\infty_V X(W) = \begin{cases} \Sigma^{W-V} X & (V \subset W) \\ * & (V \not\subset W) \end{cases}$$

spectrification

$\Sigma^\infty_V X = L\, \Pi^\infty_V X$

卡 Monad and (Operad) EKMM II 及 chapter 7

1. Monad over a Category C is a monoid in Functor(C,C) √记

自然变换

$$\begin{tikzcd} Id_c \arrow[d, "e"] & F\circ F \arrow[d, "\mu"] \\ F & F \end{tikzcd}$$

满足

$$\begin{tikzcd} F\circ F\circ F \arrow[r] \arrow[d] & F\circ F \arrow[d] \\ F\circ F \arrow[r] & F \end{tikzcd}$$

$$\begin{tikzcd} Id\circ F \arrow[r] \arrow[dr] & F\circ F \arrow[d] & F\circ Id \arrow[l] \arrow[dl] \\ & F & \end{tikzcd}$$

2. F-Algebra for a monad F

(X,ρ)

~~(X,μ,e)~~

$FX \xrightarrow[\rho]{} X$ 满足

$$\begin{tikzcd} FFX \arrow[r] \arrow[d] & FX \arrow[d] \\ FX \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "e_x"] \arrow[dr, "id"'] & FX \arrow[d] \\ & X \end{tikzcd}$$

Category 记为 $C[F]$

Example $C = Abel$, $TX = \bigoplus_{i\geq 0} X^{\otimes i}$ $PX = \bigoplus_{i\geq 0} X^{\otimes i}/\Sigma_i$

then $C[T] \simeq$ Rings $C[P] =$ Commutative Rings

(① ② ~~[illegible]~~ created in C)

3. $C \underset{Forget}{\overset{F(-)}{\rightleftarrows}} C[F]$, ① C is complete $\Rightarrow C[F]$ is complete

② C is topological ~~[illegible]~~ copowered and F continuous $\Rightarrow C[F]$ is topological powered

$f\circ h = g\circ h = id_B$

Reflexive coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend right, "h"'] \arrow[r] & C \end{tikzcd}$$

If monad $C \xrightarrow{F} C$ preserves reflexive coequalizers in C,

then ① C cocomplete $\Rightarrow C[F]$ cocomplete

$$\begin{tikzcd} F(\mathrm{colim}\, F R_\alpha) \arrow[r, shift left, "F(\mathrm{colim}\, \xi_\alpha)"] \arrow[r, shift right, "\mu\circ\alpha"'] \arrow[dr] & F(\mathrm{colim}\, R_\alpha) \arrow[r] & \mathrm{colim}_{C[F]}\, R_\alpha \\ & FF(\mathrm{colim}\, R_\alpha) \arrow[u] & \end{tikzcd}$$

$\shortparallel$

② Furthermore, C topological copowered and F continuous $\Rightarrow C[F]$ topological ~~copowered~~ complete

$(X,M) \in Top\times C[F]$ by $X \to C(M, X\otimes M) \xrightarrow{F} C(FM, F(X\otimes M))$, we get $X\otimes FM \xrightarrow{\nu} F(X\otimes M)$

$F(X\otimes FM) \underset{\mu\circ F\nu}{\overset{F(1\otimes\alpha)}{\rightrightarrows}} F(X\otimes M) \to X\otimes_{C[F]} M$ coequalizer in $C[F]$

Thm, C, D, E ~~complete~~ cocomplete, $C\times D \xrightarrow{-\boxtimes-} E$ is a functor satisfying $A\boxtimes -$, and $-\boxtimes D$ preserve colimits for any $A\in C$, $B\in D$.

then $X_1\boxtimes X_2 \rightrightarrows Y_1\boxtimes Y_2 \to Z_1\boxtimes Z_2$ is a reflexive coequalizer if $X_i \rightrightarrows Y_i \to Z_i$ are. $(i=1,2)$

Corollary: If C is a cocomplete (symmetric) closed monoidal category.

then $\underset{\substack{\cong \\ \mathrm{Mon}(C)}}{C[T]}$ and $\underset{\substack{\| \\ \mathrm{AbMon}(C)}}{C[P]}$ are cocomplete too, since T and P preserve reflexive coequalizers.