

① Stable Homotopy Groups

Freudenthal Suspension Theorem

If X is a (CGWH) space with non-degenerated basepoint,

then $\dots \pi_{i+n}(\Sigma^i X) \xrightarrow{\cong} \pi_{i+n-1}(\Sigma^{i-1} X) \dots$ is iso for $i \gg n$

$\pi_*^S \cong \varinjlim_i \pi_{*+i}(\Sigma^i X)$, $\pi_*^S(-)$ is a homology theory

but $\pi_*(-)$ is not a homology theory

② Generalized (co)homology Theory (in Top_*)

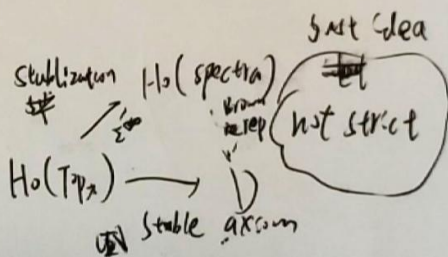
$E_n : \text{Top}_*(W^*) \rightarrow h(CW_*) \rightarrow \text{Ab}^*$

Satisfying stable axioms

1. A natural iso $h_n^*(X) \rightarrow h_{n+1}(\Sigma X)$

2. $h_n(A) \rightarrow h_n(X) \rightarrow h_n(Cf)$ is exact

3. $\bigoplus_n h_n(X_n) \xrightarrow{\cong} h_n(\bigvee_n X_n)$



So we only need to consider (co)homology theory in $\text{Ho}(\text{spectra})$

Stable Homotopy \cong Spectra

Ex. vint scale homotopy theory (LMS) 1986
 Symplectic (Klein) (cell theory)
 orthogonal spectra
 symmetric spectra (Schwede)
 vs-Cat's approach

Classical spectra

Only H_0 (Spectra)

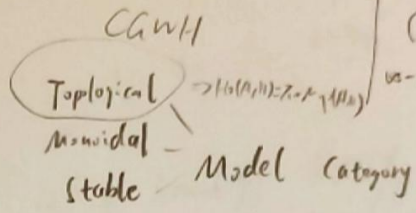
(Adams, Switzer)

$H_0(S)$ is Triangulated Cat^{*} by

of fibre sequence $X \rightarrow Y \rightarrow Cf \rightarrow \Sigma X$

$\Sigma X = X[1]$ is ~~auto~~ ^{auto equivalence}

Modern spectra



Monoidal: \circ pushout-product $A \times B \rightarrow C$
 \circ Unit $A \times B \rightarrow C$ $\Rightarrow H_0(C)$ is abelian

Stable: pointed model cat $\Rightarrow H_0(C)$ is triangulated
 \circ cofib \Rightarrow auto equivalence

Disadvantage 1. All things are ~~homotopy~~ homotopy

2. Smash product is extremely ugly, ~~unnature~~ technical.

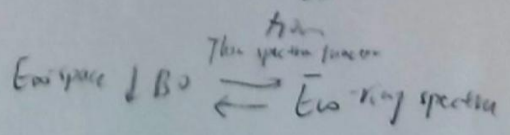
3. Homotopy module spectra cat is not stable (triangulated)

* Advantage 1. Can do something strict

2. Through monoidal model structure, we have both strict smash product^{*} Sp and homotopy smash over $H_0(S)$.

Therefore we can define Ewis space $A_{/B}$ as commutative monoid and unital object over $H_0(S)$ from strict smash product.

3. strict module spectrum cat is stable (very useful, e.g. MU-mod)



How to verify a model (category) 3

C is a category, I is a class of morphisms in C .

Def I -cof is a class of morphisms, s.t. I -fib

$$f \in I\text{-cof} \Leftrightarrow \forall g \downarrow \begin{array}{c} \xrightarrow{\quad} \\ \downarrow g \\ \xrightarrow{\quad} \end{array} \in I, \exists f' \downarrow \begin{array}{c} \xrightarrow{\quad} \\ \downarrow f' \\ \xrightarrow{\quad} \end{array}$$

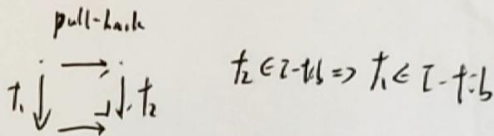
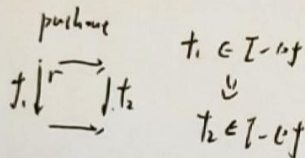
$$f \in I\text{-fib} \Leftrightarrow \forall g \downarrow \begin{array}{c} \xrightarrow{\quad} \\ \downarrow g \\ \xrightarrow{\quad} \end{array}, \exists f' \downarrow \begin{array}{c} \xrightarrow{\quad} \\ \downarrow f' \\ \xrightarrow{\quad} \end{array}$$

1. $I(I\text{-cof})\text{-fib} = I(I\text{-fib})\text{-cof}$

Prop. 1. $I \subset J \Rightarrow I\text{-cof} \supset J\text{-cof}$ and $I\text{-fib} \supset J\text{-fib}$.

2. $I\text{-cof}$ and $I\text{-fib}$ are stable under retract

3. $I\text{-cof}$ is stable under pushout, $I\text{-fib}$ is stable under pull-back

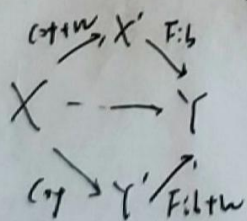
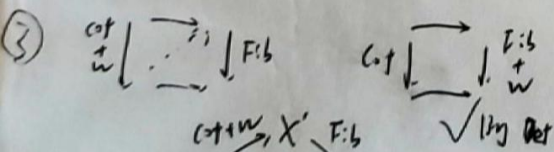


Recall: Def of model cat

$(C, \text{cof}, \text{fib}, W)$

① Bicomplete ② 2-out-of-3 for W

③ cof, fib, W are stable under retract

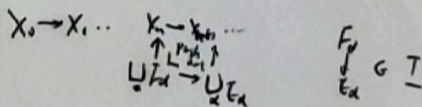


What we do usually?

1. Given a set of morphisms I , defining $\text{Fib} = I\text{-fib}$,

$\text{cof} = (\text{Fib} \cap W)\text{-cof}$. (In top, $I = \left\{ \begin{array}{c} D^{\text{no}} \\ D^{\text{no}} \times I \end{array} \right\}$)

2. prove $\lim_n C(F, X_n) \xrightarrow{\sim} C(F, \lim_n X_n) \forall F$ the domain of a morphism $f \in I$



3. By small object argument, $\forall X \rightarrow Y$, we get a factorization $X \xrightarrow{L} P \xrightarrow{R} Y$, $P \in I\text{-fib}$, $i \in (I\text{-fib})\text{-cof}$

Need prove $i \in W$

$\text{cof} = (I\text{-fib} \cap W)\text{-cof}$

Nontrivial extra inputs:

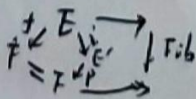
1. step 2.

2. $i \in W$ in step 3 is in W

3. step 5.

Need prove $\text{Fib} \cap W = J\text{-fib}$ for some set of morphisms (in top, $J = \left\{ \begin{array}{c} S^{\text{no}} \\ L \\ D^{\text{no}} \end{array} \right\}$)

By trivial argument, we get axiom ①



② similarly as step 3, we get axiom ②

① CGLU

② Category

EkMM's framework

4.

$$PU \rightleftharpoons SU \rightleftharpoons L\text{-Spectrum} \rightleftharpoons S\text{-module}$$

$H_*(S)$ model weakly model cat monoidal model cat

$(U \approx \mathbb{R}^\infty)$

Prospect: (\underline{A}) index set (U, A) , where U is an ^{real} inner space of countable infinite dimension

A is a ^{continuous functor} σ -final subset of $\{ \text{finite dimension} \}$ \subseteq $\{ \text{subspace of } U \}$

$$A \rightarrow \text{Top}_*$$

$V \subset W$

$$\sum_{V \subset W} E_V \rightarrow E_W \begin{cases} \text{associated} \\ \text{unit} \end{cases}$$

$E_V \rightarrow \mathcal{R}^{V \subset W} E_W$ is homeomorphism $\Leftrightarrow E$ is a spectrum

The reason \uparrow is technical (not necessarily closed)

(Subtle fact in CGLU; given a sequence of $\sqrt{\quad}$ inclusions in CGLU $X_1 \hookrightarrow X_2 \dots X_n$)

$$\lim_{\rightarrow} \mathcal{R} X_n \xrightarrow{\cong} \mathcal{R}(\lim_{\rightarrow} X_n)$$

difficulty 1

$$PA \xrightarrow{\hookrightarrow} SA$$

$$\uparrow \quad \uparrow \simeq$$

$$PB \xrightarrow{\hookrightarrow} SB$$

difficulty 2

TW : read whole - Smash product

$$SU(A \times E, E') \simeq \text{Top} \downarrow L(U, U') (A, S(E, E')) \simeq SU(E, F(A, E'))$$

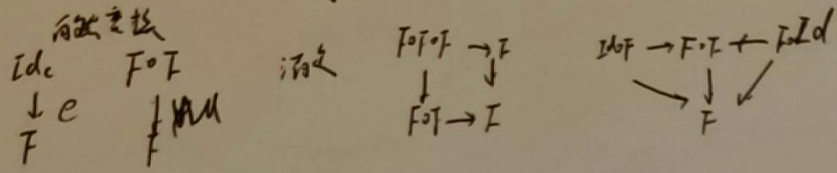
$$\sum_{V \subset W} S^V \simeq S^W$$

$$\mathbb{Z}^{\text{specification}} X(U) = \begin{cases} \sum_{V \subset W} X & (V \subset W) \\ * & (V \not\subset W) \end{cases}$$

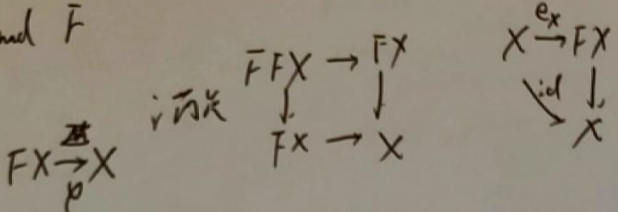
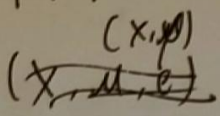
$$\sum_{V \subset W} X = \lim_{\rightarrow} \mathbb{Z}^{\text{specification}} X$$

Monad and (Operad) E(AM) 12 2 chapters

1. Monad over a Category C is a monoid in Functor(C,C) \checkmark



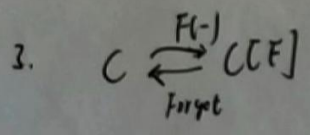
2. F-Algebra for a monad F



(category) 记为 C[F]

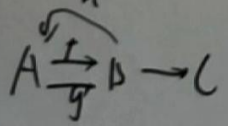
Example C = Ab, $TX = \bigoplus_{i \geq 0} X^{o_i}$ $KX = \bigoplus_{i \geq 0} X^{o_i} / \Sigma^i$

then $C[T] \approx \text{Kings}$ $C[\mu] = \text{Commutative Rings}$ (① ② ~~are~~ created in C)



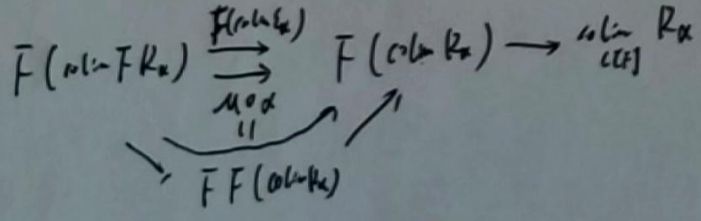
① C is complete $\Rightarrow C[F]$ is complete
 ② C is topological $\Rightarrow C[F]$ is topological provided F is topological and F continuous $f \circ h = g \circ h = \text{id}_B$

Reflexive coequalizer



If monad $\mathbb{A} C \xrightarrow{F} C$ preserves reflexive coequalizers in C,

then ① C complete $\Rightarrow C[F]$ complete



② Further \mathbb{A} C topological and F continuous $\Rightarrow C[F]$ topological ~~complete~~

$X \rightarrow C(M, X \otimes M) \xrightarrow{F} C(FM, F(X \otimes M))$, we see $X \otimes FM \xrightarrow{F} F(X \otimes M)$
 $F(X \otimes FM) \xrightarrow{F(\text{id})} F(X \otimes M) \rightarrow X \otimes M$ (equalizer in C[F])

Then, C, D, E ~~is a~~ ^{category}, $C \times D \rightarrow E$ is a functor satisfying $A \otimes -$, and $- \otimes D$ preserve objects
 for any $A \in C, B \in D$.

then $X_1 \otimes X_2 \Rightarrow Y_1 \otimes Y_2 \rightarrow Z_1 \otimes Z_2$ is a reflexive coequalizer if $X_i \Rightarrow Y_i \rightarrow Z_i$ are. (i.e., 2)

Corollary: if C is a cocomplete (symmetric) closed monoidal category,

then $\text{Coim}(C)$ and $\text{AbMon}(C)$ are complete too, since τ and ρ preserve reflexive coequalizers.

category

\bar{C}

3.

Ref

If

(2) Further
 (K, m)
 (B) π
 $\text{Top} = C[\mathbb{R}]$