An overview of ∞-categories and higher algebra

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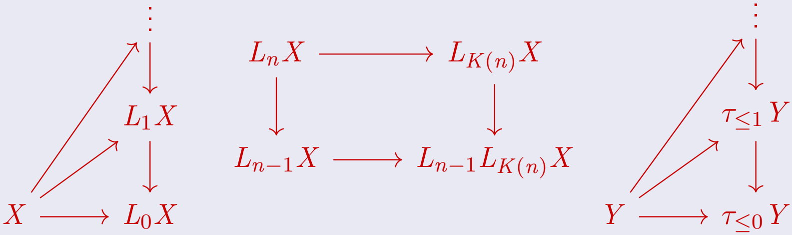
Why use ∞-categories?

Some phenomena and propositions cannot be stated in full clarity without ∞-categories.

Example

1. Chromatic convergence and chromatic pullback:



   Chromatic convergence and chromatic pullback should be described as homotopy limits of **homotopy coherent diagrams** $N(\mathbb{Z}_{\geq 0}^{op}) \to Sp$ and $\Lambda_2^2 \to Sp$ **instead of homotopy diagrams** $\mathbb{Z}_{\geq 0}^{op} \to h(Sp)$ or $\Lambda_2^2 \to h(Sp)$.

2. Similarly, a Postnikov tower in the category $\mathcal{S}$ of spaces and its convergence.

Why use ∞-categories?

Example (More)

1. If $\mathcal{C}$ is a 1-category, then $Sp(\mathcal{C}) \simeq \{*\}$ is trivial. The stabilization for 1-categories is meaningless. **Stable homotopy** is a higher categorical phenomenon.
2. By ∞-categories we can define all kinds of **moduli spaces**, such as $CAlg(Sp) \times_{CAlg(hSp)} \{R\}$, the moduli space of $\mathbb{E}_\infty$-structures on a given homotopy commutative ring spectrum R. The $\mathbb{E}_\infty$-structures on a Lubin–Tate spectrum $E(n, \Gamma)$ is **unique**, meaning $CAlg(Sp) \times_{CAlg(hSp)} \{E(n, \Gamma)\}$ is a contractible Kan complex.
3. **Bousfield localization** and **connective cover** of an $\mathbb{E}_\infty$-ring are still $\mathbb{E}_\infty$-rings. In the ∞-categorical setting, this is automatic by the fact $L_E : Sp \rightleftarrows Sp_E : i$ and $i : Sp_{\geq 0} \rightleftarrows Sp : \tau_{\geq 0}$ are symmetric monoidal adjunctions, which induce adjunctions $CAlg(Sp) \rightleftarrows CAlg(Sp_E)$ and $CAlg(Sp_{\geq 0}) \rightleftarrows CAlg(Sp)$.
4. **Equivariant stable homotopy theory**: there are numerous model categories characterizing it, but all of their underlying ∞-categories are equivalent to $Fun(BG, Sp)$, which is both simple and intuitive.

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4. **Equivariant stable homotopy theory**: there are numerous model categories characterizing it, but all of their underlying ∞-categories are equivalent to $Fun(BG, Sp)$, which is both simple and intuitive.

Motivation

The most significant motivation is to enrich the morphism **set** $Hom_{\mathcal{C}}(X, Y)$ in a category $\mathcal{C}$ to a topological **space** $Map_{\mathcal{C}}(X, Y)$. Then we can have higher morphisms $\pi_n Map_{\mathcal{C}}(X, Y)$.

For example, when considering the category of spectra, we have $\pi_n Map_{\mathcal{C}}(X, Y) = [\Sigma^n X, Y] = Y^{-n}(X)$.

So the most intuitive model for ∞-category theory should be $sSet$-enriched (or Top-enriched) categories. However, all of these models are equivalent to Joyal's model. Indeed we have Quillen equivalences $(sSet)_{Joyal} \rightleftarrows Cat_{sSet} \rightleftarrows Cat_{Top}$.

But Joyal's model encodes information more concisely: the only data of a quasi-category is a simplicial set.

Information in an ∞-category

Underlying $\mathcal{H}$-enriched category

There are many different ways to extract mapping spaces $Map_{\mathcal{C}}(X, Y)$ from an ∞-category $\mathcal{C}$.
But when we take their underlying $\mathcal{H} := Ho(sSet_{Kan})$-enriched categories, all of them become the same, written as $\underline{h\mathcal{C}}$.

Remark

The processes $\mathcal{C} \mapsto \underline{h\mathcal{C}} \mapsto h\mathcal{C}$ make it simpler to manage but meanwhile cause a loss of homotopy coherent information. How to extract useful and discard redundant information of homotopy coherence in specific circumstances is an "art" in ∞-categories' world.

Outlook and methodology

Preventing Russell's paradox

In order to consider the **category of all categories**, we need to add a set-theoretic axiom into ZFC, i.e. Grothendieck's Assumption:
$\forall$ cardinal κ, there exists an inaccessible cardinal $\tau > \kappa$. (A good reference: Chap 1, 代数学方法 1，李文威)

Methodology

By Grothendieck's Assumption,
1. When not involving **category of all categories**, technically we can treat all things as small. So all propositions not involving **category of all categories** will hold in any Grothendieck universe.
2. When involving **category of all categories**, for example $\mathcal{C}at_\infty$, we consider it as the ∞-category $\mathcal{C}at_\infty^\tau$ of all τ-small categories for an inaccessible cardinal τ. Choose a bigger inaccessible $\tau_2 > \tau$, then technically we can treat $\mathcal{C}at_\infty^\tau$ as a τ_2-small ∞-category in $\mathcal{C}at_\infty^{\tau_2}$.

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Definition (Kan extension along a full subcategory)

Let $i : \mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory, we say a functor $F : \mathcal{C} \to \mathcal{D}$ is a left Kan extension along i iff $\forall X \in \mathcal{C}$, $(\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ is a colimit diagram, i.e. $colim_{A \to X, A \in \mathcal{C}_0} F(A) \simeq F(X)$.

Theorem

The restriction $Fun^{LKan}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} Fun^{\exists LKan}(\mathcal{C}_0, \mathcal{D})$ *is a categorical equivalence.*

Example

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a category that admits small colimits, then
(1) A functor $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ is a left Kan extension along the Yoneda embedding $i : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ iff F preserves small colimits.
(2) For any $f \in Fun(\mathcal{C}, \mathcal{D})$, there exists a left Kan extension $F : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ along i.
(3) And hence we have $Fun^{colim}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$ is an equivalence. (e.g. $sSet \to Top$)

Universal properties in the category of categories

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Cocompletion

Definition

Let $\mathbb{K}$ be a collection of simplicial sets. We say that an ∞-category $\mathcal{C}$ is $\mathbb{K}$-cocomplete if it admits K-diagram colimits, for each $K \in \mathbb{K}$.

We say that a functor of ∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$ if the ∞-category $\widehat{\mathcal{C}}$ is $\mathbb{K}$-cocomplete and for every $\mathbb{K}$-cocomplete ∞-category $\mathcal{D}$, precomposition with h induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathbb{K}}(\widehat{\mathcal{C}}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

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Let $\mathbb{K}$ be a (small) collection of simplicial sets, then for any (small) ∞-category C, there exists a $\mathbb{K}$-completion $C \to P^{\mathbb{K}}(C)$. That gives an adjunction $\widehat{Cat}_\infty \rightleftarrows \widehat{Cat}(\mathbb{K})_\infty$, e.g. $P^{small}(C) = Fun(C, \mathcal{S})$ and $P^{small}(*) = \mathcal{S}$.

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More examples of universal properties

Let $\mathcal{D}$ be an ∞-category.

Theorem (Pointedlization)

If $\mathcal{D}$ admits final object, then there exists a pointedlization $\mathcal{D}_{*/} \to \mathcal{D}$ such that for any pointed ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}_*) \to \mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Final,pt} \rightleftarrows Cat_\infty^{Final}$.

Theorem (Stabilization)

If $\mathcal{D}$ admits finite limits, then there exists a stabilization $Sp(\mathcal{D}) \to \mathcal{D}$ such that for any stable ∞-category $\mathcal{C}$ the forgetful functor $\theta : \mathrm{Fun}^{Flim}(\mathcal{C}, Sp(\mathcal{D})) \to \mathrm{Fun}^{Flim}(\mathcal{C}, \mathcal{D})$ is an equivalence. That provides an adjunction $Cat_\infty^{Flim,st} \rightleftarrows Cat_\infty^{Flim}$.

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The category spectra $Sp(P(*))$ is the stabilization of the cocompletion of the trivial ∞-category.

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Let $n \geq -2$, an object Z in an ∞-category C is n-truncated if, for every object $Y \in C$, the space $Map_C(Y, Z)$ is n-truncated space.

Theorem (Truncation)

If C is a presentable ∞-category, then there exists an n-truncation functor $C \to \tau_{\leq n} C$. Suppose that $\mathcal{D}$ is a presentable that all objects are n-truncated, i.e. it's an $(n+1)$-category. Then composition with $\tau_{\leq n}$ induces an equivalence $s : \mathrm{Fun}^{\mathrm{L}}(\tau_{\leq n}\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$. That provides an adjunction $Pr^L \rightleftarrows Pr^L_{\leq (n+1)}$.

Example

(1) An space X in $\mathcal{S}$ is n-truncated iff all $\pi_i X$ vanish when $i > n$. Particularly $\mathcal{S}_{\leq 0} \simeq N(Set)$.

(2) An n-truncated object Cat_∞ is exactly an n-category. And all n-categories form an $(n+1)$-category $(Cat_\infty)_{\leq n}$.

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Postnikov-type tower

Let $\mathcal{C}$ be an ∞-category and $I = \{\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_n \subset \cdots \subset \mathcal{C}\}$ be an ascending sequence of reflective full subcategories of $\mathcal{C}$, where **reflective** means the inclusion functor $\mathcal{C}_i \hookrightarrow \mathcal{C}$ admits a left adjoint.

Example

1. If taking $I = \{\mathcal{S}_{\leq 0} \subset \mathcal{S}_{\leq 1} \subset \cdots \subset \mathcal{S}_{\leq n} \subset \cdots \subset \mathcal{S}\}$, we recover to the classical case.
2. If taking $I = \{L_0 Sp^{\omega}_{(p)} \subset L_1 Sp^{\omega}_{(p)} \subset \cdots \subset L_n Sp^{\omega}_{(p)} \subset \cdots \subset Sp^{\omega}_{(p)}\}$ where $Sp^{\omega}_{(p)}$ is the ∞-category of finite p-local spectra, we get chromatic convergence case.

Definition (Tower and pretower)

1. An I-tower in $\mathcal{C}$ is a functor $\mathrm{N}(\mathbb{Z}^{op}_{\geq 0})^{\triangleleft} \to \mathcal{C}$, which we view as a diagram $X_\infty \to \cdots \to X_2 \to X_1 \to X_0$ satisfying that for each $n \geq 0$, the map $X_\infty \to X_n$ exhibits X_n as a $\mathcal{C}_n$-reflection of X_∞.
2. An I-pretower in $\mathcal{C}$ is a functor $\mathrm{N}(\mathbb{Z}^{op}_{\geq 0}) \to \mathcal{C}$: $\cdots \to X_2 \to X_1 \to X_0$ which exhibits each X_n as a $\mathcal{C}_n$-reflection of X_{n+1}.

Postnikov-type convergence

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Definition

We let $\mathrm{Post}_I^+(\mathcal{C})$ denote the ∞-category of I-towers, and $\mathrm{Post}_I(\mathcal{C})$ the ∞-category of I-pretowers. We have an evident forgetful functor $\phi : \mathcal{C} \xleftarrow{\sim} \mathrm{Post}_I^+(\mathcal{C}) \to \mathrm{Post}_I(\mathcal{C})$. We will say that $\mathcal{C}$ is **Postnikov I-complete** if ϕ is an equivalence of ∞-categories.

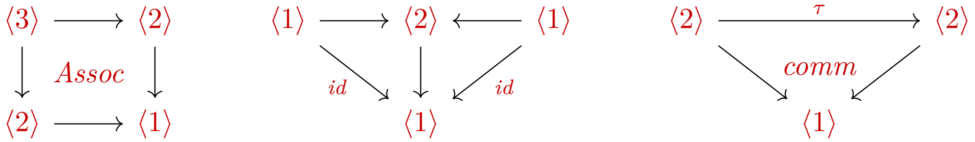
Theorem (Postnikov-type convergence)

Suppose that any I-pretower in $\mathcal{C}$ has a limit. Then $\mathcal{C}$ is Postnikov I-complete
if and only if,
for every diagram $X : \mathrm{N}(\mathbb{Z}_{\geq 0}^{op})^{\triangleleft} \to \mathcal{C}$ the following conditions are equivalent:
(1) The diagram X is an I-tower.
(2) The diagram X is a limit in $\mathcal{C}$, and the restriction of X to $\mathrm{N}(\mathbb{Z}_{\geq 0}^{op})$ is an I-pretower.

Higher commutative monoids

Definition (Reformulation of ordinary commutative monoids)

A (3-)commutative monoid in an ordinary category $\mathcal{C}$ which admits finite products is a functor $M : (Fin_*)_{\leq 3} \to \mathcal{C}$ such that the canonical maps $M(\rho_i) : M(\langle n \rangle) \to M(\langle 1 \rangle)$ exhibit $M(\langle n \rangle) \simeq \prod_{1 \leq i \leq n} M(\langle 1 \rangle)$ in $\mathcal{C}$ for all $0 \leq n \leq 3$.



Definition (∞-commutative monoid)

Let $\mathcal{C}$ be an ∞-category with finite products. We define an ∞-commutative monoid in $\mathcal{C}$ as a functor $M : N(Fin_*) \to \mathcal{C}$ such that the canonical maps $M(\rho_i) : M(\langle n \rangle) \to M(\langle 1 \rangle)$ exhibit $M(\langle n \rangle) \simeq \prod_{1 \leq i \leq n} M(\langle 1 \rangle)$ in $\mathcal{C}$ for all $n \geq 0$.

Symmetric monoidal ∞-category

Proposition (Barkan 2022)

Let $\mathcal{C}$ be a complete n-category. Then $CMon^{\infty}(\mathcal{C}) \xrightarrow{\sim} CMon^{n+2}(\mathcal{C})$ is categorically equivalent.

Definition

A symmetric monoidal ∞-category is an (∞-)commutative monoid in Cat_{∞}.

Corollary

Particularly, if a symmetric monoidal ∞-category $\mathcal{C}$ is a 1-category, then it is an ∞-commutative monoid in $(Cat_{\infty})_{\leq 1}$, which is a 2-category and written as $Cat_{\leq 1}$. So we have $CMon^{\infty}(Cat_{\leq 1}) \xrightarrow{\sim} CMon^{4}(Cat_{\leq 1})$.

It can be checked that the 4-commutativity in $Cat_{\leq 1}$ exactly corresponds with ordinary coherent conditions of a symmetric monoidal category.

Lurie's definition

By the (un)straightening equivalence $Fun(N(Fin_*), Cat_\infty) \simeq CoCart_{/N(Fin_*)}$, we get the following equivalent definition by Lurie.

Definition

A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p : \mathcal{C}^\otimes \to N(Fin_*)$ with the following property:
For each $n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$ induce functors $\rho^i_! : \mathcal{C}^\otimes_{\langle n \rangle} \to \mathcal{C}^\otimes_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^\otimes_{\langle n \rangle} \simeq (\mathcal{C}^\otimes_{\langle 1 \rangle})^n$.
We define $\mathcal{C}^\otimes_{\langle 1 \rangle}$ as its underlying ∞-category.

This definition has technical advantages for general ∞-operads.

Tensor product of ∞-categories

Let $\mathbb{K}$ be the collection of all small simplicial sets.

Definition

Given 2 cocomplete ∞-categories C and D, we define the tensor product as a functor $C \times D \to C \otimes D$ such that for any cocomplete E, we have $Fun^{\mathbb{K}}(C \otimes D, E) \xrightarrow{\sim} Fun^{\mathbb{K}\boxtimes\mathbb{K}}(C \times D, E)$. Such tensor product always exists because the natural functor $C \times D \to \mathcal{P}^{\mathbb{K}}_{\mathbb{K}\boxtimes\mathbb{K}}(C \times D)$ satisfies that.

Theorem

The above gives a symmetric monoidal structure $\widehat{Cat}_\infty(\mathbb{K})^{\otimes} \to N_*(Fin_*)$ and makes the cocompletion funcor a symmetric monoidal adjunction $\widehat{Cat}_\infty^{\otimes} \rightleftarrows \widehat{Cat}_\infty(\mathbb{K})^{\otimes}$. So $S = \mathcal{P}(*)$ is the unit in $\widehat{Cat}_\infty(\mathbb{K})^{\otimes}$.

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Cocomplete symmetric monoidal structure

Remark

By (un)straightening equivalence, $CAl(\widehat{Cat}_\infty(\mathbb{K})) \subset CAl(\widehat{Cat}_\infty)$ is the subcategory whose objects are symmetric monoidal ∞-categories such that $- \otimes -$ preserves colimits separately in each variable (called **cocomplete symmetric monoidal** categories), and whose morphisms are **colimit-preserving** symmetric monoidal functors.

Corollary

The symmetric monoidal adjunction induces an adjunction between algebras $F : CAl(\widehat{Cat}_\infty) \rightleftarrows CAl(\widehat{Cat}_\infty(\mathbb{K}))$.

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(1) The $\mathcal{S} = \mathcal{P}(*)$ is the unit in $\widehat{Cat}_\infty(\mathbb{K})^\otimes$, which means it is initial object in $CAl(\widehat{Cat}_\infty(\mathbb{K}))$ and hence $\mathcal{S}$ admits a cocomplete symmetric monoidal structure $\mathcal{S}$.
(2) So for any cocomplete symmetric monoidal ∞-category, there exists essentially unique colimit-preserving symmetric monoidal functor $\mathcal{S}^\otimes \to \mathcal{C}^\otimes$.

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Localization

Proposition (Localization)

Let $\mathcal{C}$ be an ∞-category and let $L : \mathcal{C} \to \mathcal{C}$ be a functor with essential image $L\mathcal{C} \subseteq \mathcal{C}$. The following conditions are equivalent:
(1) There exists a functor $f : \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $g : \mathcal{D} \to \mathcal{C}$ and an equivalence between $g \circ f$ and L.
(2) When regarded as a functor from $\mathcal{C}$ to $L\mathcal{C}$, L is a left adjoint of the inclusion $L\mathcal{C} \subseteq \mathcal{C}$.
(3) There exists a natural transformation from $\mathrm{id}_{\mathcal{C}} \to L$ such that, $L \circ \mathrm{id}_{\mathcal{C}} \to L \circ L$ and $\mathrm{id}_{\mathcal{C}} \circ L \to L \circ L$ are equivalences in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, i.e. an idempotent object in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

Proposition

The full subcat $Pr^L \subset \widehat{Cat}_\infty(\mathbb{K})$ is closed under tensor product and hence inherits a symmetric monoidal structure $Pr_L^{\otimes}$.

Symmetric monoidal colocalization

Proposition (Symmetric monoidal colocalization)

*Let $\mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal ∞-category. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory which is stable under equivalence. Suppose that the functor $-\otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ carries $\mathcal{D} \times \mathcal{D}$ into $\mathcal{D}$ (meaning $\mathcal{D}$ is **closed under tensor products**). Then the following hold.*

1. *The restricted map $\mathcal{D}^{\otimes} \to N(Fin_*)$ is a symmetric monoidal ∞-category.*
2. *The inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a symmetric monoidal functor.*
3. *Suppose that the inclusion $\mathcal{D} \subseteq \mathcal{C}$ admits a right adjoint L (so that $\mathcal{D}$ is a colocalization of $\mathcal{C}$). Then there exists a lax-symmetric-monoidal right adjunction $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$.*

Formally speaking, $L^{\otimes}$ is a right adjunction in the strict 2-category $h_2(Op_{/\mathcal{O}^{\otimes}})$.

Corollary

Under assumptions of (3) above, a symmetric monoidal colocalization can induce a colocalization on algebras $CAlg(\mathcal{D}) \rightleftarrows CAlg(\mathcal{C})$.

Connective cover

Corollary (t-structure and symmetric monoidal structure)

Let $p : \mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal ∞-category. Assume that the underlying ∞-category $\mathcal{C}$ is stable and that $- \otimes -$ is exact in each variable. We will say that a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is **compatible** with the symmetric monoidal structure if the functor $\otimes$ carries $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$.

Then the induced map $\mathcal{C}^{\otimes}_{\geq 0} \to N(Fin_*)$ is again a symmetric monoidal ∞-category, and $\mathcal{C}^{\otimes}_{\geq 0} \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} \mathcal{C}^{\otimes}$ is a symmetric monoidal colocalization. Thus it further induces a colocalization $CAlg(\mathcal{C}_{\geq 0}) \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} CAlg(\mathcal{C})$.

Example (Connective cover of an $\mathbb{E}_\infty$-ring)

When $\mathcal{C} = Sp$ we have $CAlg(Sp_{\geq 0}) \underset{\tau_{\geq 0}}{\overset{i}{\rightleftarrows}} CAlg(Sp)$, which means that the connective cover of an $\mathbb{E}_\infty$-ring naturally inherits an $\mathbb{E}_\infty$-structure.

Symmetric monoidal localization

Proposition (Symmetric monoidal localization)

Let $\mathcal{C}^{\otimes} \to N(Fin_*)$ be a symmetric monoidal ∞-category. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Suppose that $\mathcal{D} \subset \mathcal{C}$ is a reflective subcategory (with a left adjoint $L : \mathcal{C} \to \mathcal{D}$). If for every pair g_1, g_2 of L-equivalences in $\mathcal{C}$, the morphism $g_1 \otimes g_2$ in $\mathcal{C}$ is also an L-equivalence (meaning ***L-equivalences are closed under tensor products****), then we have the following.*

1. *The restricted map $\mathcal{D}^{\otimes} \to N(Fin_*)$ is lax-symmetric-monoidal.*
2. *The inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a symmetric monoidal functor.*
3. *There exists a symmetric monoidal left adjoint $L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$.*

Corollary

A symmetric monoidal localization can induce a localization on algebras $CAlg(\mathcal{C}) \rightleftarrows CAlg(\mathcal{D})$.

Bousfield localization

Let $\mathcal{C}^{\otimes}$ be a presentably symmetric monoidal ∞-category, i.e., an object in $CAlg(Pr^L) \hookrightarrow CAlg(Cat_{\infty})$.

Theorem (Bousfield localization)

Let $E \in \mathcal{C}$ be an object. Then $W_E = \{X \to Y | X \otimes E \xrightarrow{\sim} Y \otimes E\} \subset Fun(\Delta^1, \mathcal{C})$ is a small-generated strongly saturated collection of morphisms, which means that there exists an accessible localization functor $L_E : \mathcal{C} \to \mathcal{C}$.

Furthermore, Bousfield localization is compatible with its symmetric monoidal structure, meaning it forms a symmetric monoidal localization $\mathcal{C}^{\otimes} \underset{i^{\otimes}}{\overset{L_E^{\otimes}}{\rightleftarrows}} \mathcal{C}_E^{\otimes}$.

Example (Bousfield localization of an $\mathbb{E}_{\infty}$-ring)

When $\mathcal{C} = Sp$ we have $CAlg(\mathcal{C}) \underset{i}{\overset{CAlg(L_E)}{\rightleftarrows}} CAlg(\mathcal{C}_E)$. This means that Bousfield localization of an $\mathbb{E}_{\infty}$-ring naturally inherits an $\mathbb{E}_{\infty}$-structure.

Idempotent object

Let $\mathcal{C}$ be a symmetric monoidal ∞-category.

Definition (idempotent object)

Let $e : 1_C \to E$ be a morphism in $\mathcal{C}$. We say e is idempotent iff $1_\mathcal{C} \otimes X \to X \otimes X$ is equivalent. (e.g. $\mathbb{Z} \to \mathbb{Z}[1/p]$ in Ab)

Theorem (Bousfield localization with respect to an idempotent object)

Let $e : 1_C \to E$ be a morphism in $\mathcal{C}$, then
(1) The e is an idempotent object of $\mathcal{C}$ iff the transformation $\alpha : \mathrm{id}_\mathcal{C} \to l_E$ exhibits l_E as a localization functor on $\mathcal{C}$, where $l_E : \mathcal{C} \to \mathcal{C}$ is given by the tensor product with E.
(2) If e is idempotent, then l_E is exactly the Bousfield localization with respect to E, which has the following properties:

(a) The l_E is compatible with $\otimes$, so induces a symmetric monoidal localization

$$\mathcal{C}^\otimes \underset{i^\otimes}{\overset{L_E^\otimes}{\rightleftarrows}} \mathcal{C}_E^\otimes \ ;$$

(b) The inclusion $i^\otimes$ is also symmetric monoidal, meaning $\mathcal{C}_E$ is closed under tensor products.

Idempotent algebra

Definition

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We will say that a commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$ is idempotent if unit map $e : \mathbf{1} \to A$ is idempotent.

Theorem

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$, which we regard as a trivial algebra object of $\mathcal{C}$. Then the functor

$$\theta : \mathrm{CAlg}^{idem}(\mathcal{C}) \subseteq \mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})_{1/} \to \mathcal{C}_{\mathbf{1}/}$$

is fully faithful, and its essential image are idempotent objects in $\mathcal{C}$, which gives an equivalence $\mathrm{CAlg}^{idem}(\mathcal{C}) \xrightarrow{\sim} (\mathcal{C}_{\mathbf{1}/})^{idem}$.
Furthermore, any mapping space in $(\mathcal{C}_{\mathbf{1}/})^{idem}$ is either empty or contractible, i.e. $(\mathcal{C}_{\mathbf{1}/})^{idem}$ is equivalent to a partial-order set $N(I)$.

Interesting applications after the internalization

Proposition

The full subcat $Pr^L \subset \widehat{Cat}_\infty(\mathbb{K})$ *is closed under tensor products* ($\mathcal{S}$ *is also the unit in* Pr^L) *and hence inherits a symmetric monoidal structure. In fact, for any* $\mathcal{C}, \mathcal{D} \in Pr^L$, *we have a natural equivalence* $\mathcal{C} \otimes D \simeq RFun(\mathcal{C}^{op}, \mathcal{D})$.

Theorem (Unique symmetric monoidal structure)

The following 4 colimit-preserving functors $\mathcal{S} \xrightarrow{\tau_{\leq n}} \tau_{\leq n}\mathcal{S}$, $\mathcal{S} \xrightarrow{(-)_+} \mathcal{S}_*$, $\mathcal{S} \xrightarrow{\Sigma^\infty_+} Sp$, *and* $\mathcal{S} \xrightarrow{* \mapsto \mathbb{Z}} N(Ab)$ *are idempotent objects in* Pr^L.
Hence by $\mathrm{CAlg}(Pr^L)^{idem} \xrightarrow{\sim} (Pr^L_{\mathcal{S}/})^{idem}$ *we conclude that* $\mathcal{S}$ *resp.* $\mathcal{S}_{\leq n}$, $\mathcal{S}_*$, Sp, $N(Ab)$ *only admits a unique cocomplete symmetric monoidal structure with the unit* $*$ *resp.* $*$, S^0, $\Sigma^\infty S^0$, $\mathbb{Z}$.

Interesting applications after the internalization

By Bousfield localization with respect to idempotent objects, we have:

Corollary

The following 4 full subcategories of Pr^L are closed under tensor products.
(a) $Pr^L_{\leq n+1}$: the ∞-category of presentable $(n+1)$-categories;

(b) Pr^L_* : the ∞-category of presentable pointed ∞-categories;

(c) Pr^L_{st} : the ∞-category of presentable stable ∞-categories, known as tensor-triangulated ∞-categories or tt-∞-categories;
(d) $Pr^L_{1\text{-}ad}$: the ∞-category of presentable additive 1-categories.

Corollary

The localization functors $Pr^L \xrightarrow{-\otimes \tau_{\leq n}\mathcal{S}} Pr^L_{\leq n+1}$, $Pr^L \xrightarrow{-\otimes \mathcal{S}_*} Pr^L_*$, $Pr^L \xrightarrow{-\otimes Sp} Pr^L_{st}$, and $Pr^L \xrightarrow{-\otimes N(Ab)} Pr^L_{1\text{-}ad}$ correspond with the n-truncation, copointedlization, costabilization, and 1-coadditivalization of presentable ∞-categories respectively.

My specific interests

Use higher algebra and spectral algebraic geometry (SAG) to explore various intersections between homotopy theory and algebraic geometry.

Example

1. A recent good example is the Chromatic Nullstellensatz by Burklund, Schlank, and Yuan. They proved that "algebraically closed" $\mathbb{E}_\infty$-rings in $CAlg(Sp_{T(n)})$ are exactly those Lubin–Tate spectra $E(L)$ with L an algebraically closed field. And for any non-zero $T(n)$-local $\mathbb{E}_\infty$-ring R, there exists a geometric point $R \to E(L)$.
2. For another beautiful example, the Devinatz–Hopkins theorem $L_{K(n)}S \simeq E_n^{h\mathbb{G}_n}$ can be interpreted as $\mathrm{QCoh}(\mathrm{Spf}(E_n)/\mathbb{G}_n) \simeq Sp_{K(n)}$ in (formal) SAG.
3. In the framework of SAG, we can study spectral moduli problems: given an algebro-geomtric stack $\mathcal{M}_0$, can we give an $\mathbb{E}_\infty$-realization $\mathcal{M}$ making $\pi_0\mathcal{M} = \mathcal{M}_0$?
   It is true when $\mathcal{M}_0 = \mathcal{M}_{ell}$ for the moduli stack of elliptic curves and when $\mathcal{M}_0 = \mathcal{X}_{K^p}$ for some of the Shimura stacks. Then taking the global sections of $\mathbb{E}_\infty$-stacks respectively, we get TMF and TAF, which are intriguing $\mathbb{E}_\infty$-rings.

My specific interests

Orientation theory from Thom spectra

1. Quillen discovered a deep connection between (homotopy) complex orientation set $\mathrm{Or}_h(MU, E) := \mathrm{Hom}_{CAlg(hSp)}(MU, E)$ and formal group laws over E_*, which became the cornerstone of chromatic homotopy theory.
2. After that, Ando–Hopkins–Strickland discovered a correspondence $\mathrm{Or}_h(MU\langle 6\rangle, E) \xrightarrow{g_3} C^3(P_E; \mathcal{I}(0))$ between $MU\langle 6\rangle$-orientations and cubical structures. By uniqueness of cubical structures on any line bundle of any abelian variety, we can endow a unique $MU\langle 6\rangle$-orientation to any elliptic cohomology theory.

$\mathbb{E}_\infty$-enhancement of orientations

When comes to $\mathbb{E}_\infty$-orientation space $\mathrm{Or}_{\mathbb{E}_\infty}(Mf, R) := \mathrm{Map}_{CAlg(Sp)}(Mf, R)$, combining the Thom adjunction $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})_{/Pic(Sp)} \underset{}{\overset{M(-)}{\rightleftarrows}} CAlg(Sp)$ and the infinite loop space machine $\mathrm{Mon}^{gp}_{\mathbb{E}_\infty}(\mathcal{S}) \simeq Sp_{\geq 0}$ we can produce many interesting results.

My specific interests

$$\mathcal{S}p_{\geq 0} \xrightarrow{\sim} \mathrm{Mon}^{gp}_{\mathbb{E}_\infty}(\mathcal{S}) \underset{GL_1}{\leftrightarrows} \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathrm{CAlg}(\mathcal{S}p)$$

$$gl_1 : \mathrm{CAlg}(\mathcal{S}p) \to \mathcal{S}p_{\geq 0}$$

By this adjunction we can get the following theorem.

Theorem (Ando–Blumberg–Gepner–Hopkins–Rezk)

Let Mf *be the Thom* $\mathbb{E}_\infty$*-spectrum induced by a map* $f : X \to \mathrm{pic}(\mathcal{S}p)$ *in* $\mathcal{S}p_{\geq 0}$ *and let* R *be an* $\mathbb{E}_\infty$*-ring. Then* $\mathrm{Or}_{\mathbb{E}_\infty}(Mf, R)$ *is a torsor over the H-space* $\mathrm{Map}_{\mathcal{S}p}(X, gl_1(R))$*, meaning* $\mathrm{Or}_{\mathbb{E}_\infty}(Mf, R)$ *is either empty or homotopy equivalent to* $\mathrm{Map}_{\mathcal{S}p}(X, gl_1(R))$*.*

Example

Particularly, combining with the Chromatic Nullstellensatz and some further calculations, we can deduce that for any $\text{height} = n > 0$, the $\mathrm{Or}_{\mathbb{E}_\infty}(MUP, E(\overline{\mathbb{F}}_p))$ is non-empty and hence homotopy equivalent to $\mathrm{Map}_{\mathcal{S}p}(ku, gl_1(E(\overline{\mathbb{F}}_p)))$.