Let E be an EWP.

① Let C be a category having finite products. $\qquad F_K(x,y) = x+y-v\cdot xy$

If $X, Y \in CMon(C)$, then $C^t(X,Y)$ is well-defined for any $t \geq 0$

② For any $X \in Ho(Top)$, $E^0(P^t)\hat{\otimes}_{E_0} E_0(X) \xrightarrow{\simeq} Hom_{E_0}(E_0P^t, E_0X)$

③ If X is an even commutative H-space, then for $t \geq 0$ we have

$$\begin{tikzcd}
C^t(P,X) \arrow[r] & C^t_{E_0\text{-}CoAlg}(E_0P, E_0X) \arrow[r, dashed] \arrow[d] \arrow[dr, dashed] & Hom_{Mon_E}(X^E, \underline{C}^t(P_E, G_{m,E})) \arrow[d] \\
& \underline{C}^t(P_E, M_{m,E})(\mathrm{spec}\, E_0X) & \underline{C}^t(P_E, G_{m,E})(X^E) \arrow[l]
\end{tikzcd}$$

④ $t \geq 1$

$$\begin{tikzcd}
C^t(P, BU\langle 2t\rangle) \arrow[r, hook, "\text{subgroup}"] \arrow[d] & bu^{2t}(P^t) \arrow[d, hook] \\
C^t(P, BU\langle 0\rangle) \arrow[r, hook] & bu^0(P^t)
\end{tikzcd}$$

⑤
$$\begin{tikzcd}
BU\langle 2t\rangle^E \arrow[r, "\Psi_t"] \arrow[dr] & \underline{C}^t(P_E, G_m) \arrow[d] \\
& S^E
\end{tikzcd}$$
is a homomorphism of S^E-group schemes. for $t \geq 0$

⑥ $V_* P_{t+1} = \delta(P_t)$ for $t \geq 1$

$$\begin{tikzcd}
\rho_t \in C^t(P, BU\langle 2t\rangle) \arrow[r] & C^{t+1}(P, BU\langle 2t\rangle) \\
C^{t+1}(P, BU\langle 2t+2\rangle) \ni \rho_{t+1} \arrow[ur]
\end{tikzcd}$$

⑦
$$\begin{tikzcd}
BU\langle 2t\rangle^E \arrow[r, "V^E"] \arrow[d] & BU\langle 2t+2\rangle^E \arrow[d] \\
\underline{C}^t(P_E, G_{m,E}) \arrow[r, "\delta"] & \underline{C}^{t+1}(P_E, G_{m,E})
\end{tikzcd}$$
commutes for $t \geq 1$

⑧ $\widetilde{E^0(P^k)^\vee} \in Pic(\mathrm{spf}\, E^0P^k)$ $\qquad \forall k \geq 1$ we have $MU\langle 2k\rangle^E \to \underline{C}^k(P_E, L)$, $\quad L = \widetilde{E^0(P^k)}$

$\forall k \geq 1$

$$\begin{tikzcd}
BU\langle 2k\rangle^E \times MU\langle 2k\rangle^E \arrow[r] \arrow[d] & \underline{C}^k(P_E, G_m) \times \underline{C}^k(P_E, L) \arrow[d] \\
MU\langle 2k\rangle^E \arrow[r] & \underline{C}^k(P_E, L)
\end{tikzcd}$$

$\forall k \geq 1$

$$\begin{tikzcd}
MU\langle 2k\rangle^E \arrow[r] \arrow[d] & C^k(P_E, L) \arrow[d, "\delta"] \\
MU\langle 2k+2\rangle^E \arrow[r] & \underline{C}^{k+1}(P_E, L)
\end{tikzcd}$$

⑨ $\forall$ Emp ring spectrum morphism $E \to F$

$k \geq 1$

$$\begin{tikzcd}
BU\langle 2k\rangle^F \arrow[r] \arrow[d] & \underline{C}^k(P_F, G_m) \arrow[r] \arrow[d] & S^F \arrow[d] \\
BU\langle 2k\rangle^E \arrow[r] & \underline{C}^k(P_E, G_m) \arrow[r] & S^E
\end{tikzcd}$$

⑩ Let $f \in E_0$, then $E[f^{-1}]$ is a phantom ring spectrum.

And $E[f^{-1}] \leftarrow MP$ and $E \to E[f^{-1}]$ are phantom ring morphisms.

We have $E[f^{-1}]^0(P) \xrightarrow{\simeq} \varprojlim Hom_{E_0[f^{-1}]}(E[f^{-1}]_0(P_n), E[f^{-1}]_0)$

So
$$\begin{tikzcd}
BU\langle 2k\rangle^{MP} \arrow[r] & \underline{C}^k(P_{MP}, G_m) \\
BU\langle 2k\rangle^{E[f^{-1}]} \arrow[r] \arrow[u] \arrow[d] & \underline{C}^k(P_{E[f^{-1}]}, G_m) \arrow[u] \arrow[d] \\
BU\langle 2k\rangle^E \arrow[r] & C^k(P_E, G_m)
\end{tikzcd}$$

Thom spectrum adjunction

$$\mathrm{Top}_{/BF} \underset{G_F}{\overset{\mathrm{Th}(-)}{\rightleftarrows}} \mathrm{Sp} \qquad G_F = EF \times_F \Omega^\infty(-)$$

$$\Omega^\infty(-) \longrightarrow EF \times_F \Omega^\infty(-) \longrightarrow BF$$

$$\mathrm{Top}[E_\infty]_{/BF} \overset{\mathrm{Th}(-)}{\rightleftarrows} \mathrm{Sp}[E_\infty]$$

$$\begin{tikzcd} BU\langle 6\rangle \arrow[r] \arrow[d] & BO\langle 8\rangle \arrow[d] \\ BSU \arrow[r] \arrow[d] & BSpin \arrow[d] \\ BU \arrow[r] \arrow[dr] & BSO \arrow[d] \\ & BF \quad BO \arrow[l] \end{tikzcd}$$

in $\mathrm{Top}[E_\infty]_{/BF}$ $\Rightarrow$

$$\begin{tikzcd} MU\langle 6\rangle \arrow[r] \arrow[d] & MO\langle 8\rangle = MString \arrow[d] \\ MSU \arrow[r] \arrow[d] & MSpin \arrow[d] \\ MU \arrow[r] \arrow[dr] & MSO \arrow[d] \\ & MF \quad MO \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} F = (\Omega^\infty \Sigma^\infty S^0)^* \arrow[r] \arrow[d] & QS^0 \arrow[d] \\ \{\pm 1\} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Infinite loop space machine

$$\mathrm{Top}[E_\infty] \underset{\Omega^\infty}{\overset{\Sigma_f = \Sigma^\infty \otimes_{K_*}(-)}{\rightleftarrows}} \mathrm{Sp} \qquad K_* \xrightarrow{f} \Omega^\infty\Sigma^\infty \text{ in } \mathrm{Fun}(\mathrm{Top}, \mathrm{Top})$$

$$\mathrm{Ho}(\text{group-like } \mathrm{Top}[E_\infty]) \overset{\sim}{\rightleftarrows} \mathrm{Ho}(\mathrm{Sp}_{\geq 0})$$

$$[MU\langle 6\rangle, E] \xrightarrow{\sim} C^3(P_E, I(0))$$

Cal(hcp)

$$\begin{tikzcd} & E \arrow[d] \\ MU\langle 6\rangle \arrow[ur] \arrow[r] & F \end{tikzcd} \qquad \begin{tikzcd} (E, C_1) \arrow[d] \\ (F, C_2) \end{tikzcd}$$

$$\mathrm{Ho}(\mathrm{Sp}_{\geq 0}) \underset{\Omega^\infty}{\overset{\Sigma_f}{\underset{\sim}{\rightleftarrows}}} \text{group-like } \mathrm{Ho}(\mathrm{Top}[E_\infty]) \underset{GL_1}{\overset{i}{\rightleftarrows}} \mathrm{Ho}(E_\infty\text{-space}) \underset{\Omega^\infty_\times}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Sp}[E_\infty])$$

①

Cart H subspace

$$\begin{tikzcd} GL_1 X \arrow[r] \arrow[d] & X \arrow[d] \\ \pi_0(X)^* \arrow[r] & \pi_0 X \end{tikzcd}$$

$\mathrm{Ho}(\mathrm{Top})$-enriched functors

$$\mathrm{Ho}(\mathrm{Sp}_{\geq 0}) \underset{gl_1}{\overset{\Sigma^\infty_+ \Omega^\infty(-)}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Sp}[E_\infty])$$

$$gl_1 = g \circ \Omega^\infty_\times = \Sigma^f GL_1 \Omega^\infty_\times$$

$$[MString, R]_{E_\infty} = \pi_0 \mathrm{Map}_{E_\infty}(MString, R)$$

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Sp}[E_\infty]}(MString, R) \simeq \mathrm{Map}_{\mathrm{Top}[E_\infty]_{/BF}}(BString, G_F(R)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Top}[E_\infty]}(BString, G_F(R)) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{Top}[E_\infty]}(BString, BF) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Sp}[E_\infty]}(MString, R) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Sp}}(bstring, g(G_F(R))) \arrow[d] \\ * \arrow[r] & \mathrm{Map}_{\mathrm{Sp}}(bstring, bF) \end{tikzcd}$$

$$bstring = \Sigma^f BString$$

$$bF = \Sigma^f BF$$

$$
\begin{tikzcd}
 & & \Sigma gl_1 S \arrow[d, "\wr"] \arrow[dr, "\Sigma gl_1(-)"] & \\
gl_1 R \arrow[r] & g(G_F(R)) \arrow[r] & bF \arrow[r] & \Sigma gl_1 R \\
 & & bstring \arrow[u] \arrow[ur, "k"'] \arrow[ul, dashed, "?"] &
\end{tikzcd}
$$

$k=0$

If $k=0$, then choose a lifting $\alpha : bstring \to g(G_F(E))$

we get $Map_{E_\infty}(Mstring, R) \simeq Map_{Sp}(bstring, gl_1(R))$

$Map_{E_\infty}(Mstring, R)$ is a $Map_{sp}(bstring, gl_1(R))$-torsor

$$
\begin{tikzcd}
\pi_0 Map_{E_\infty}(Mstring, R) \arrow[rr, dashed] \arrow[d, "\simeq"] & & \prod_{k\geq 2} (\pi_{4k}(R) \otimes \mathbb{Q}) = [bstring, gl_1 R \otimes \mathbb{Q}] \arrow[d, "\wr"] \\
\pi_0 Map_{E_\infty / BF}(Bstring, G_F(R)) \arrow[r, "\pi_0 \Omega^\infty"'] & \pi_0 Map_{sp}(bstring, g(G_F(R))) \arrow[r] & \pi_0 Map_{sp}(bstring, gG_F(R) \otimes \mathbb{Q}) = [bstring, gG_F(R) \otimes \mathbb{Q}]
\end{tikzcd}
$$

since $gl_1 S \otimes \mathbb{Q} \simeq 0$

$$(b_8, b_{12}, b_{16} \cdots) \in \prod_{k \geq 2} \pi_{4k}(R) \otimes \mathbb{Q}$$

$$F \to B(*,F,F) \to B(*,F,*)$$

$$\begin{tikzcd} g(F) \arrow[r] \arrow[d, equal] & g(EF) \arrow[r] & g(BF) \arrow[r, "\sim"] & \Sigma gl_1 S \\ gl_1 S & & & \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[r] \arrow[d] & B(*,F,F) \arrow[r] \arrow[d] & B(*,F,*) \arrow[d, "\wr"] \\ GL_1 R \arrow[r] & B(*,F,GL_1 R) \arrow[r] & B(*,F,*) \end{tikzcd}$$

$$\begin{tikzcd} gl_1 S \arrow[r] \arrow[d] & g(EF) \arrow[r] \arrow[d, "\wr"] & g(BF) \arrow[r, "\sim"] \arrow[d, "\wr"] & \Sigma gl_1 S \arrow[d] \\ gl_1(R) \arrow[r] & g\, G_F(R) \arrow[r] & g(BF) \arrow[r] & \Sigma gl_1 R \end{tikzcd}$$

$X, E \in CAlg(hSp)$

if E_*E is flat E_*-fl

then E_*X is a E_*E-comodule E_*-Algebra

$TM_0 = N_*(\mathcal{E}(A))$

ob $\mathcal{E}(A) = \{X \mid E_*X \simeq A\}$ Mor $\mathcal{E}(A)$ $X_1 \xrightarrow{E_\infty} X_2$

such that $E_*X_1 \xrightarrow{\sim} E_*X_2$

$TM_0 \to TM_0 \sim E_*E$-comodule E_*-Algebra

$\exists\ X \xrightarrow{f} E_n$ s.t.

$$\begin{tikzcd} E_*X \arrow[r, "\varphi"] & E_*E \arrow[d] \\ & E_* \end{tikzcd}$$

$f : X \to E$

$E_*f : E_*X \xrightarrow{\varphi} E_*E$

$$\begin{tikzcd} \mathrm{Hom}_{E_*}(E_*E_n, E_*E) \arrow[r, "\varphi^*", "\sim"'] & \mathrm{Hom}_{E_*}(E_*X, E_*E) \\ {[E_n, E]} \arrow[u, "E_*"] \arrow[r, dashed] & {[X, E]} \arrow[u, "\wr"] \end{tikzcd}$$

id_X

$id \mapsto f$

$$\begin{tikzcd} \pi_*E\wedge X \arrow[d, "\varphi"'] \arrow[d, shift left, "E_*f"] \arrow[dr, "id\wedge p"] & \\ \pi_*E\wedge E_n \arrow[r, "id"] & \pi_*E \end{tikzcd}$$

Let $\mathcal{C}$ be the category of commutative and cocommutative Hopf algebra over a field k.

⓪ $\mathcal{C}$ is additive, and $\mathcal{C}$ has kernels and cokernels.

承认: ① Newman Thm: $\forall H \in \mathcal{C}$,

$$\left\{ \begin{matrix} \text{Hopf ideals} \\ \text{of } H \end{matrix} \right\} \underset{I(G) = HG^+}{\overset{G(I) = \mathrm{Hker}(H \to H/I)}{\overset{\cong}{\rightleftarrows}}} \left\{ \begin{matrix} \text{Hopf subalgebra} \\ \text{of } H \end{matrix} \right\}$$

② Any injective morphism in $\mathcal{C}$ is a monomorphism, any surjective morphism is an epimorphism.

②.⑤ $\mathrm{Hker}(X \xrightarrow{f} Y) = \mathrm{Hker}(X \to X/\ker(f)) = \sigma(\ker f)$

③ Any surjective morphism is a cokernel. ④ Any monomorphism is injective, and hence a kernel.

⑤ Any epimorphism is surjective, and hence a cokernel. ⑥ $\mathcal{C}$ is an abelian Category!

⑦ monomorphism $\Leftrightarrow$ injective, epimorphism $\Leftrightarrow$ surjective.