Copointedlization and costabilization

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Introduction

This note is motivated by the following statement that appeared in the introduction part of [1], which I quote here:

"... let us consider for a moment the central role played by the ∞-category $\mathrm{Sp}_{\mathrm{fin}}$ of finite spectra in the theory of stable ∞-categories. To begin, $\mathrm{Sp}_{\mathrm{fin}}$ can be described as the free stable ∞-category generated by a single object $\mathbb{S} \in \mathrm{Sp}_{\mathrm{fin}}$, the sphere spectrum. Furthermore, one can use $\mathrm{Sp}_{\mathrm{fin}}$ in order to characterize stable ∞-categories inside the ∞-category $\mathrm{Cat}_{\mathrm{fin}}$ of all small ∞-categories with finite colimits (and right exact functors between them). Indeed, $\mathrm{Cat}_{\mathrm{fin}}$ carries a natural symmetric monoidal structure (see [2, §4.8.1]) whose unit is the the smallest full subcategory of spaces $\mathcal{S}_{\mathrm{fin}} \subseteq \mathcal{S}$ closed under finite colimits. One can then show that $\mathrm{Sp}_{\mathrm{fin}}$ is an idempotent object in $\mathrm{Cat}_{\mathrm{fin}}$ in the following sense: the suspension spectrum functor $\Sigma^\infty_+ : \mathcal{S}_{\mathrm{fin}} \longrightarrow \mathrm{Sp}_{\mathrm{fin}}$ induces an equivalence $\mathrm{Sp}_{\mathrm{fin}} \simeq \mathrm{Sp}_{\mathrm{fin}} \otimes \mathcal{S}_{\mathrm{fin}} \xrightarrow{\simeq} \mathrm{Sp}_{\mathrm{fin}} \otimes \mathrm{Sp}_{\mathrm{fin}}$. The fact that $\mathrm{Sp}_{\mathrm{fin}}$ is idempotent has a remarkable consequence: it endowed $\mathrm{Sp}_{\mathrm{fin}}$ with a canonical commutative algebra structure in $\mathrm{Cat}_{\mathrm{fin}}$ such that the forgetful functor $\mathrm{Mod}_{\mathrm{Sp}_{\mathrm{fin}}}(\mathrm{Cat}_{\mathrm{fin}}) \longrightarrow \mathrm{Cat}_{\mathrm{fin}}$ is fully-faithful. From a conceptual point of view, this fact can be described as follows: given an ∞ category with finite colimits $\mathcal{C}$, the structure of being an $\mathrm{Sp}_{\mathrm{fin}}$-module is essentially unique once it exists, and can hence be considered as a property. One can then show that this property coincides with being stable.

In other words, stable ∞ categories are exactly those $\mathcal{C} \in \mathrm{Cat}_{\mathrm{fin}}$ which admit an action of $\mathrm{Sp}_{\mathrm{fin}}$, in which case the action is essentially unique."

The main goal of this note is to prove the following theorem.

Theorem 0.1 (costabilization). *1. The inclusion $\mathrm{Cat}_{st} \subset \mathrm{Cat}_{\mathrm{fin}}$ is reflective. And the natural functor $\mathcal{C} \to \mathrm{colim}(\Sigma : \mathcal{C} \to \mathcal{C})$ is a reflection when $\mathcal{C}$ is pointed. We will denote $\mathrm{cSp}(\mathcal{C})$ as $\mathrm{colim}(\Sigma : \mathcal{C} \to \mathcal{C})$.*

2. The natural transformation $\mathcal{C} \to \mathcal{C} \otimes \mathrm{Sp}_{\mathrm{fin}}$ is also a reflection to the inclusion $\mathrm{Cat}_{st} \subset \mathrm{Cat}_{\mathrm{fin}}$, where the tensor product is the bilinear finite cocompletion in $\mathrm{Cat}^{\otimes}_{\mathrm{fin}}$. That implies $\mathcal{S}_{\mathrm{fin}} \to \mathrm{Sp}_{\mathrm{fin}}$ is idempotent in $\mathrm{Cat}^{\otimes}_{\mathrm{fin}}$.

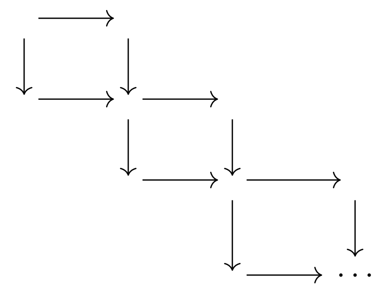
1. Copointedlization

Definition 1.1. *Let $\mathcal{A}$ be an ∞-category with finite colimits. We say a functor $\mathcal{A} \to \mathcal{B}$ is a copointedlization if $\mathcal{B}$ is a pointed ∞-category with finite colimits and $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{B}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{A}, \mathcal{C})$ is an equivalence for any pointed ∞-category $\mathcal{C}$ with finite colimits.*

Corollary 1.2. *The full subcategory $\mathrm{Cat}_{\mathrm{fin},*} \subset \mathrm{Cat}_{\mathrm{fin}}$ of pointed ∞-categories with finite colimits is reflective.*

2. A concrete model of $\mathrm{cSp}(\mathcal{C})$

Definition 2.1. *We denote $I = \bigvee_{n=0}^{\infty} \Delta^1 \times \Delta^1$ as the following diagram*



Let $\mathcal{C}$ be a pointed ∞-category with finite colimits. We denote $\mathrm{Fun}^{\Sigma}(I, \mathcal{C}) \subset \mathrm{Fun}(I, \mathcal{C})$ as the full subcategory spanned by those diagrams in which every square is a suspension pushout.

Proposition 2.2. *Let $\mathcal{C}$ be a pointed ∞-category with finite colimits. Then the upper left corner evaluation functor $\mathrm{Fun}^{\Sigma}(I, \mathcal{C}) \xrightarrow{ev_0} \mathcal{C}$ is an equivalence.*

Proof. It is not hard to see the evaluation functor $\mathrm{Fun}^{\Sigma}(I_{\leq n}, \mathcal{C}) \xrightarrow{ev_0} \mathcal{C}$ is an equivalence. Then by $\mathrm{Fun}^{\Sigma}(I, \mathcal{C}) = \varprojlim_n \mathrm{Fun}^{\Sigma}(I_{\leq n}, \mathcal{C})$ the result follows. □

Lemma 2.3. *(see HTT 5.5.7.11.) Let κ be a regular cardinal. Then* $\mathrm{Cat}_{\mathrm{Rex}(\kappa)} \to \mathrm{Cat}_\infty$ *preserves κ-filtered colimits.*

Definition 2.4. 1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a right exact functor where $\mathcal{C}$ is a pointed ∞-category with finite colimits and $\mathcal{D}$ is a stable ∞-category. We say F is a costabilization if* $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{C}, \mathcal{E})$ *is an equivalence for any stable ∞-category $\mathcal{E}$.*

2. *We define* $\mathrm{cSp}(\mathcal{C}) = \varinjlim_n (\Sigma : \mathrm{Fun}^\Sigma(I, \mathcal{C}) \to \mathrm{Fun}^\Sigma(I, \mathcal{C})) \in \mathrm{Cat}_{\mathrm{fin}}$*, where the Σ is induced by the evaluation on subdiagram removing first square. By the lemma above the colimit can be formulated in the category of simplicial sets.*

Theorem 2.5. *Let $\mathcal{C}$ be a pointed ∞-category with finite colimits. Then the functor* $\mathcal{C} \xrightarrow{\sim} \mathrm{Fun}^\Sigma(I, \mathcal{C}) \to \mathrm{cSp}(\mathcal{C})$ *is a costabilization.*

Proof. We first claim that $\mathrm{cSp}(\mathcal{C})$ is stable. Actually the suspension functor of $\mathrm{cSp}(\mathcal{C})$ has the form $\varinjlim_n \mathrm{Fun}^\Sigma(I, \mathcal{C}) \to \varinjlim_{n+1} \mathrm{Fun}^\Sigma(I, \mathcal{C})$, which is an equivalence obviously. So $\mathrm{cSp}(\mathcal{C})$ is stable.
Now given a stable ∞-category $\mathcal{E}$, we note that $\mathrm{Fun}^{\mathrm{Rex}}(\mathrm{cSp}(\mathcal{C}), \mathcal{E}) = \varprojlim_n \mathrm{Fun}^{\mathrm{Rex}}(\mathrm{Fun}^\Sigma(I, \mathcal{C}), \mathcal{E})$. However $\mathrm{Fun}^{\mathrm{Rex}}(\mathrm{Fun}^\Sigma(I, \mathcal{C}), \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{Rex}}(\mathcal{C}, \mathcal{E})$ is stable. And functors in the limit diagram can all be identified as the suspension functor of $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{C}, \mathcal{E})$, so this is a constant limit and we are done. □

Corollary 2.6. *The full subcategory* $\mathrm{Cat}_{st} \subset \mathrm{Cat}_{\mathrm{fin},*}$ *is reflective.*
Hence $\mathrm{Cat}_{st} \subset \mathrm{Cat}_{\mathrm{fin},*} \subset \mathrm{Cat}_{\mathrm{fin}}$ *is reflective by 1.2.*

Remark 2.7. *Although the copointedlization coincides with the pointedlization, it seems there is no reason that* $\mathrm{cSp}(\mathcal{C}) = \varinjlim(\Sigma : \mathcal{C} \to \mathcal{C})$ *would be equal to the* $\mathrm{Sp}(\mathcal{C}) = \varprojlim(\Sigma : \mathcal{C} \to \mathcal{C})$ *appeared in [2] for an arbitrary $\mathcal{C}$.*

References

[1] Y. Harpaz. Ambidexterity and the universality of finite spans. *Proceedings of the London Mathematical Society*, 121(5):1121–1170, July 2020. (document)

[2] J. Lurie. Higher algebra. Sept. 2017. (document), 2.7