∞-topoi and parametrized homotopy theory

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1-topoi

Proposition

Let $\mathcal{C}$ be a category. The following conditions are equivalent:

1. *The category $\mathcal{C}$ is (equivalent to) the category of sheaves $Sh(X)$ of sets on some Grothendieck site X.*
2. *The category $\mathcal{C}$ is (equivalent to) a left exact localization of the category $PSh(\mathcal{C}_0)$ of presheaves of sets on some small category $\mathcal{C}_0$.*
3. *Giraud's axioms are satisfied:*
   1. *The category $\mathcal{C}$ is presentable (that is, $\mathcal{C}$ has small colimits and a set of small generators).*
   2. *Colimits in $\mathcal{C}$ are universal.*
   3. *Coproducts in $\mathcal{C}$ are disjoint.*
   4. *Equivalence relations in $\mathcal{C}$ are effective.*

Definition (1-topos)

If a $\mathcal{C}$ satisfies the equivalent conditions above, we call it a (1-)topos.

Why we need ∞-topoi

1. As the basis of unstable homotopy theory.

Example

- (i) The ∞-category of spaces $\mathcal{S}$ is the basic but also most important example of ∞-topos.
- (ii) Also the ∞-category of G-spaces $\mathcal{S}_G$ is an ∞-topoi.
- (iii) Although the ∞-category of motivic spaces $H(S)$ for a Noetherian scheme S is not an ∞-topos, the Nisnevich sheaf involves lots of ∞-topos techniques.

2. As the basis of parametrized homotopy theory.
3. As the basis of spectral algebraic geometry.

Definition of ∞-topoi

Proposition

Let $\mathcal{X}$ be an ∞-category. The following conditions are equivalent:

1. *The ∞-category X is an ∞-topos: i.e. if there exists a small ∞-category $\mathcal{C}$ and an accessible left exact localization functor $\mathcal{P}(\mathcal{C}) \to \mathcal{X}$.*
2. *The ∞-category X is presentable, and colimits in which are universal, i.e. $(\operatorname{colim} X_\alpha) \times_Z Y \simeq \operatorname{colim}(X_\alpha \times_Z Y)$. And furthermore it satisfies that $\mathcal{X}_{/X} \simeq \lim \mathcal{X}_{/X_\alpha}$ when $X = \operatorname{colim} X_\alpha$.*
3. *The ∞-category $\mathcal{X}$ satisfies the following ∞-categorical analogues of Giraud's axioms:*
   1. *The ∞-category X is presentable.*
   2. *Colimits in $\mathcal{X}$ are universal.*
   3. *Coproducts in X are disjoint.*
   4. *Every groupoid object of X is effective.*

Note that an ∞-topos is no longer necessarily the ∞-category of sheaves on a Grothendieck topology! And we will be discussing that later.

Homotopy theory in an ∞-topos

Since every ∞-topos is a left localization of some presheaf ∞-category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$, it shares lots of properties upon the $\mathcal{S}$.

Lemma

For an ∞-topos $\mathcal{X}$, $\tau_{\leq n}\mathcal{X} \subset \mathcal{X}$ is stable under finite products.

Definition (homotopy groups)

Let $f : X \to Y$ be a morphism in an ∞-topos $\mathcal{X}$. Regarding f as an object of the topos $\mathcal{X}_{/Y}$, we may take its 0-truncation $\tau_{\leq 0}^{\mathcal{X}_{/Y}} f$. This is a discrete object of $\mathcal{X}_{/Y}$, and we define $\pi_0(f) \simeq f^* \tau_{\leq 0}^{\mathcal{X}_{/Y}}(X) \simeq X \times_Y \tau_{\leq 0}^{\mathcal{X}_{/Y}}(f)$ in $\tau_{\leq 0}(\mathcal{X}_{/X})$.
If $n > 0$, then we define $\pi_n(f) \simeq \pi_{n-1}(\delta)$, where $\delta : X \to X \times_Y X$ is the associated diagonal map.

We can identify $\delta^n(f) = (X \to X^{S^{n-1}})$ in $\mathcal{X}_{/Y}$, which makes $\pi_n(f)$ is a group object in the ordinary topos $\tau_{\leq 0}(\mathcal{X}_{/X})$ when $n \geq 1$ and an abelian group object when $n \geq 2$ by the lemma above.

Homotopy groups

Remark

If $\mathcal{X} = \mathcal{S}$ and $\eta : * \to X$ is a pointed space, then $\eta^* \pi_n(X)$ can be identified with the nth homotopy group of X with base point η.

Proposition

Let $f : X \to Y$ be an n-truncated morphism in an ∞-topos $\mathcal{X}$. Then $\pi_k(f) \simeq *$ for all $k > n$. If furthermore $n \geq 0$ and $\pi_n(f) \simeq *$, then f is $(n-1)$-truncated.

Proposition

Given a pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an ∞-topos $\mathcal{X}$, there is an natural exact sequence of pointed objects

$$\cdots \to f^* \pi_{n+1}(g) \xrightarrow{\delta_n} \pi_n(f) \to \pi_n(g \circ f) \to f^* \pi_n(g) \xrightarrow{\delta_n} \pi_{n-1}(f) \to \cdots$$

in the ordinary topos $\mathrm{Disc}\left(\mathcal{X}_{/X}\right)$.

n-connective

Definition

Let $\mathcal{C}$ be a presentable ∞-category and $n \geq -2$. We define $(n+1)\text{-conn} = {}^{\perp}(n\text{-trun})$, meaning a morphism is $(n+1)$-connective iff it is left orthogonal with all n-truncated morphisms.

Proposition

For any presentable ∞-category $\mathcal{C}$ and any $n \geq -2$, the pair $((n+1)\text{-conn}, n\text{-trun})$ is a factorization system.

Proposition

Let $f : X \to Y$ be a morphism in an ∞-topos $\mathcal{X}$. Then

1. *Every morphism f in $\mathcal{X}$ is (-1)-connective.*
2. *Let $0 \leq n \leq \infty$. Then f is n-connective iff it is an effective epimorphism and $\pi_k(f) = *$ for $0 \leq k < n$. We shall say that an object X is n-connective if $f : X \to 1_{\mathcal{X}}$ is n-connective, where $1_{\mathcal{X}}$ denotes the final object of X.*

∞-connective and hypercomplete

Whitehead theorem does not necessarily hold for every ∞-topos, because there could exist non-trivial ∞-connective morphisms.

Proposition

Let $\mathcal{X}$ be an ∞-topos and let S denote the collection of ∞-connective morphisms of $\mathcal{X}$. Then S is strongly saturated, stable under pullback and of small generation.

We denote the $\hat{\mathcal{X}}$ as the left exact localization by inverting all ∞-connective morphisms, which is also an ∞-topos.

Definition

Let $\mathcal{X}$ be an ∞-topos. We say that it is hypercomplete if every ∞-connective morphism of $\mathcal{X}$ is an equivalence.

Proposition

Let $\mathcal{X}$ be an ∞-topos. Then the hypercompletion $\hat{\mathcal{X}}$ is a hypercomplete ∞-topos.

Grothendieck topology

Definition (Sieve)

1. Let $\mathcal{C}$ be an ∞-category. A sieve on $\mathcal{C}$ is a full subcategory of $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ having the property that if $f : C \to D$ is a morphism in $\mathcal{C}$, and D belongs to $\mathcal{C}^{(0)}$, then C also belongs to $\mathcal{C}^{(0)}$.
2. Let $\{X_\alpha\}$ be a collection of objects in $\mathcal{C}$. Then we can associate a sieve $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ by $\mathcal{C}^{(0)} = \{X \in \mathcal{C} | \exists X \to X_\alpha \text{ for some } \alpha\}$, which is the smallest sieve containing $\{X_\alpha\}$.
3. If $X \in \mathcal{C}$ is an object, then a sieve on X is a sieve on the ∞-category $\mathcal{C}_{/X}$. Given a morphism $f : X \to Y$ and a sieve $\mathcal{C}^{(0)}_{/Y}$ on Y, we let $f^*\mathcal{C}^{(0)}_{/Y}$ denote the sieve on X such that $f^*\mathcal{C}^{(0)}_{/Y} \subseteq \mathcal{C}_{/X}$ and a morphism $A \to X$ is in $f^*\mathcal{C}^{(0)}_{/Y}$ iff the composition $A \to X \to Y$ is in $\mathcal{C}^{(0)}_{/Y}$.

Grothendieck topology

Definition

A Grothendieck topology on an ∞-category $\mathcal{C}$ consists of a specification, for each object C of $\mathcal{C}$, of a collection of sieves on C which we will refer to as covering sieves. The collections of covering sieves are required to possess the following properties:

1. If C is an object of $\mathcal{C}$, then the $\mathcal{C}_{/C}$ itself is a covering sieve on C.
2. If $f : C \to D$ is a morphism in $\mathcal{C}$ and $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on D, then $f^*\mathcal{C}^{(0)}_{/C}$ is a covering sieve on C.
3. Let C be an object of $\mathcal{C}, \mathcal{C}^{(0)}_{/C}$ a covering sieve on C, and $\mathcal{C}^{(1)}_{/C}$ an arbitrary sieve on C. Suppose that, for each $f : D \to C$ belonging to the sieve $\mathcal{C}^{(0)}_{/C}$, the pullback $f^*\mathcal{C}^{(1)}_{/C}$ is a covering sieve on D. Then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

Proposition

For an ∞-category $\mathcal{C}$, the collection of Grothendieck topologies on $\mathcal{C}$ is naturally bijective to that on the 1-category $\mathrm{N}(\mathrm{h}\mathcal{C})$.

Grothendieck topology

Example

Let X be a topological space and $\mathcal{U}(X)$ be the partially ordered set of all open subsets of X, which can be endowed with the Zariski (etale, smooth or fppf) Grothendieck topology by that a sieve $\mathcal{U} \subset \mathcal{U}(X)_{/U}$ on U is a covering sieve iff it is generated by a collection of Zariski (etale, smooth or fppf) morphisms $\{U_\alpha \to U\}$ with $U = \bigcup U_\alpha$.

Sieves and monomorphisms

For each object $U \in \mathcal{P}(\mathcal{C})$, let $\mathcal{C}^{(0)}(U) \subseteq \mathcal{C}$ be the full subcategory spanned by those objects $C \in \mathcal{C}$ such that $U(C) \neq \emptyset$. It is easy to see that $\mathcal{C}^{(0)}(U)$ is a sieve on $\mathcal{C}$. Conversely, given a sieve $\mathcal{C}^{(0)} \subseteq \mathcal{C}$, there is a unique map $\mathcal{C} \to \Delta^1$ such that $\mathcal{C}^{(0)}$ is the preimage of $\{0\}$. This construction determines a bijection between sieves on $\mathcal{C}$ and functors $f : \mathcal{C} \to \Delta^1$, and we may identify $\Delta^1 \subset \mathcal{S}^{op}$ as the full subcategory spanned by the objects $\emptyset, \Delta^0 \in \mathcal{S}^{op}$. Since every (-1)-truncated Kan complex is equivalent to either $\emptyset$ or Δ^0, we conclude:

Proposition

For every small ∞-category $\mathcal{C}$, the construction $U \mapsto \mathcal{C}^{(0)}(U)$ determines an equivalence $Sie(\mathcal{C}) \simeq \tau_{\leq -1}\mathcal{P}(\mathcal{C})$ of partially order sets between (-1)-truncated objects of $\mathcal{P}(\mathcal{C})$ and of all sieves on $\mathcal{C}$. Furthermore, this bijection preserves the inclusion relation, so we have a natural equivalence of partially order sets $Sie(\mathcal{C}) \simeq \tau_{\leq -1}\mathcal{P}(\mathcal{C})$.

Corollary

We have the following equivalence $Sie(\mathcal{C}_{/X}) \simeq \tau_{\leq -1}\mathcal{P}(\mathcal{C}_{/X}) \simeq \tau_{\leq -1}\mathcal{P}(\mathcal{C})_{/X}$, where the latter exactly corresponds with all monomorphisms to X in $\mathcal{P}(\mathcal{C})$.

∞-Sheaves and Topological localization

Definition (sheaf)

Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology. Let S be the collection of all monomorphisms $U \to j(C)$ which correspond to covering sieves $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$. An object $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ is a sheaf if it is S-local. We let $\mathrm{Shv}(\mathcal{C})$ denote the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by S-local objects.

Proposition

A presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ is a sheaf iff $(\mathcal{C}^{(0)}_{/C})^{\triangleright} \to \mathcal{C} \xrightarrow{\mathcal{F}^{op}} \mathcal{S}^{op}$ is a colimit for any covering sieve $\mathcal{C}^{(0)}_{/C}$.

Definition (topological localization)

Let $L : \mathcal{C} \to \mathcal{D}$ be an accessible left exact localization of presentable ∞-categories. Then we say that it is topological if the strongly saturated class of those f sending to an equivalence by L is generated by a collection of monomorphisms.

(co)topological decomposition of an ∞-topos

Theorem

Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology. Then $\mathrm{Shv}(\mathcal{C})$ is a topological localization of $\mathcal{P}(\mathcal{C})$. In particular, $\mathrm{Shv}(\mathcal{C})$ is an ∞-topos.

Definition

Let $L : \mathcal{C} \to \mathcal{D}$ be an accessible left exact localization of presentable ∞-categories. Then we say that it is cotopological (or homotopical) if it satisfies that any f sending to an equivalence by L is ∞-connective.

Theorem

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}'' \subseteq \mathcal{X}$ be an accessible left exact localization of $\mathcal{X}$. Then there exists a unique topological localization $\mathcal{X}' \subseteq \mathcal{X}$ such that $\mathcal{X}'' \subseteq \mathcal{X}'$ is a cotopological localization of $\mathcal{X}'$.

(co)topological decomposition of an ∞-topos

By this unique decomposition, we see that every ∞-topos $\mathcal{X}$ can be obtained in following way:

1. Begin with the ∞-category $\mathcal{P}(\mathcal{C})$ of presheaves on some small ∞-category $\mathcal{C}$.
2. Choose a Grothendieck topology on $\mathcal{C}$: this is equivalent to choosing a left exact localization of the underlying topos $\mathrm{Disc}(\mathcal{P}(\mathcal{C})) = \mathrm{Set}^{\mathrm{h}\mathcal{C}^{op}}$.
3. Form the associated topological localization $\mathrm{Shv}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$, which can be described as the pullback
$$\mathcal{P}(\mathcal{C}) \times_{\mathcal{P}(\mathrm{N}(\mathrm{h}\mathcal{C}))} \mathrm{Shv}(\mathrm{N}(\mathrm{h}\mathcal{C}))$$
in $\mathcal{R}\mathcal{T}opoi$.
4. Form a cotopological localization of $\mathrm{Shv}(\mathcal{C})$ by inverting some subclass of ∞-connective morphisms of $\mathrm{Shv}(\mathcal{C})$.

Remarks about (co)topological localization

The hypercompletion $\hat{\mathcal{X}}$ is, in some sense, at the other extreme: it is obtained by inverting the ∞-connective morphisms in $\mathcal{X}$, which are never monomorphisms unless they are already equivalences. In fact, $\hat{\mathcal{X}}$ is the maximal left exact localization of $\mathcal{X}$ which can be obtained without inverting monomorphisms:

Proposition

Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi and let $f^* : \mathcal{X} \to \mathcal{Y}$ be a left exact colimit-preserving functor. The following conditions are equivalent:

1. *For every monomorphism u in $\mathcal{X}$, if f^*u is an equivalence in $\mathcal{Y}$, then u is an equivalence in $\mathcal{X}$.*
2. *For every morphism $u \in \mathcal{X}$, if f^*u is an equivalence in y, then u is ∞-connective.*

Parametrized homotopy theory

Definition ($\mathcal{C}$-valued sheaves on a Grothendieck topology)

Let $\mathcal{T}$ be an small ∞-category equipped with a Grothendieck topology. Let $\mathcal{C}$ be an arbitrary ∞-category. Similar to sheaves valued on spaces, we will say that a functor $\mathcal{O} : \mathcal{T}^{\mathrm{op}} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf on $\mathcal{T}$ if the following condition is satisfied: for every object $U \in \mathcal{T}$ and every covering sieve $\mathcal{T}^0_{/U} \subseteq \mathcal{T}_{/U}$, the composite map

$$\left(\mathcal{T}^0_{/U}\right)^{\triangleleft} \subseteq \left(\mathcal{T}_{/U}\right)^{\triangleleft} \to \mathcal{T} \xrightarrow{\mathcal{O}^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}$$

is a colimit diagram in $\mathcal{C}^{\mathrm{op}}$. We let $\mathrm{Shv}_{\mathcal{C}}(\mathcal{T}) \subset \mathrm{Fun}\left(\mathcal{T}^{\mathrm{op}}, \mathcal{C}\right)$ denote the full subcategory of spanned by the $\mathcal{C}$-valued sheaves on $\mathcal{T}$.

Definition ($\mathcal{C}$-valued sheaves on an ∞-topos)

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C}$ be an arbitrary ∞-category. A $\mathcal{C}$-valued sheaf on $\mathcal{X}$ is a functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ which preserves small limits. We let $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ spanned by the $\mathcal{C}$-valued sheaves on $\mathcal{X}$.

Comparison of 2 definitions

Proposition

Let $\mathcal{T}$ be a small ∞-category equipped with a Grothendieck topology. Let $j : \mathcal{T} \to \mathcal{P}(\mathcal{T})$ denote the Yoneda embedding and $L : \mathcal{P}(\mathcal{T}) \to \mathcal{S}hv(\mathcal{T})$ a left adjoint to the inclusion. Let $\mathcal{C}$ be an arbitrary ∞-category which admits small limits. Then composition with $L \circ j$ induces an equivalence of ∞-categories $\mathrm{Shv}_{\mathcal{C}}(\mathrm{Shv}(\mathcal{T})) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{T})$.

proof: It follows from the composition $\mathrm{Shv}_{\mathcal{C}}(\mathrm{Shv}(\mathcal{T})) \to \mathrm{Fun}^{\lim}(\mathcal{P}(\mathcal{T})^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{C})$ is fully faithful, and its essential image is the full subcategory $\mathrm{Shv}(\mathcal{T})$.

Remark

Let $\mathcal{C}$ be a presentable ∞-category and $\mathcal{X}$ an ∞-topos. Then the ∞ category $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ can be identified with the tensor product $\mathcal{C} \otimes \mathcal{X}$ introduced in § HA.4.8.1 . In particular, $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ is a presentable ∞-category.

Example

1. Parametrized over a space:
   Let $\mathcal{C}$ be a ∞-category and let T be an ∞-groupoid. The ∞-category $\mathrm{Fun}(T^{\mathrm{op}}, \mathcal{C})$ of $\mathcal{C}$-valued presheaves on T is naturally equivalent to the ∞-category $\mathrm{Shv}(\mathcal{S}_{/T})$ of $\mathcal{C}$–valued sheaves on $\mathcal{S}_{/T}$ via the natural equivalences
   $$\mathrm{Fun}(T^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}^{\lim}(\mathcal{P}(T)^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}^{\lim}(\mathcal{S}_{/T}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{S}_{/T}).$$
2. Parametrized over a compact Lie group (equivariant homotopy):
   For a compact Lie group G, we set $\mathcal{S}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{C})$ as the category of G-spaces, where $\mathcal{O}_G$ is the orbit category of G. This is a (presheaf) ∞-topos.

   Recall that the full subcategory of (CGWH)-spaces on the homogeneous G-spaces, that is Hausdorff spaces with a transitive G-action, is equivalent to the full subcategory spanned by the orbits G/H, where $H \leq G$ is a closed subgroup. By $\mathcal{O}_G$ we denote the associated ∞-category which we call the orbit category of G.

   For any ∞-category $\mathcal{C}$, we call an object in $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{S}_G)$ by a G-object.

Spectrum-valued sheaves on an ∞-topos

Definition

Let $\mathcal{X}$ be an ∞-topos. A sheaf of spectra on $\mathcal{X}$ is a sheaf on $\mathcal{X}$ with values in the ∞-category Sp of spectra. We let $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}\left(\mathcal{X}^{\mathrm{op}}, \mathrm{Sp}\right)$ spanned by the sheaves of spectra on $\mathcal{X}$.

Proposition

By identification $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \simeq \mathrm{Sp} \otimes \mathcal{X}$, we conclude that it is a stable ∞-category and that the natural functor $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{X}$ represents as the stabilization of $\mathcal{X}$.

Stabilization of an ∞-topos

Definition

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\heartsuit} = \tau_{\leqslant 0}\mathcal{X}$ denote its underlying topos. Composing the forgetful functor functor $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$ with the truncation $\mathcal{X} \to \tau_{\leqslant 0}\mathcal{X}$, we obtain a functor $\pi_0 : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \tau_{\leqslant 0}\mathcal{X}$.

More generally, for any integer n, we let $\pi_n : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \tau_{\leqslant 0}\mathcal{X}$ denote the composition of the functor π_0 with the shift functor $\Omega^n : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$.

Note that π_n preserves finite products and that $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is stable. It follows that π_n can be regarded as a functor from $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ to the category of abelian groups objects of $\mathcal{X}^{\heartsuit}$.

Lemma

If $\mathcal{C}$ is a 1-topos, then the category of its abelian groups objects $\mathrm{Ab}(\mathcal{C})$ is a Grothendieck abelian category, meaning that it is presentable and that monomorphisms in it are closed under small filtered colimits.

Definition

For every integer n, the functor $\Omega^{\infty-n}:\mathrm{Sp}\to\mathcal{S}$ induces a functor $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})\to\mathrm{Shv}_{\mathcal{S}}(\mathcal{X})\simeq\mathcal{X}$, which we will also denote by $\Omega^{\infty-n}$. We will say that an object $\mathcal{F}\in\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is n-truncated if $\Omega^{\infty+n}\mathcal{F}$ is a discrete object of $\mathcal{X}$. We will say that a sheaf of spectra $\mathcal{F}\in\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is n-connective if the homotopy groups $\pi_m\mathcal{F}$ vanish for $m<n$.

We will say that M is connective if it is 0-connective (equivalently, M is connective if the object $\Omega^{\infty-m}\mathcal{F}\in\mathcal{X}$ is m-connective for every $m\geq 0$). We let $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant n}$ denote the the full subcategory of $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ spanned by the n-connective objects, and $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant n}$ the full subcategory of $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ spanned by the n-truncated objects.

Stable homotopy theory on an ∞-topos

Theorem

Let $\mathcal{X}$ be an ∞-topos.

1. *The full subcategories $(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant 0}, \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0})$ determine a t-structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$.*
2. *The t-structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is compatible with filtered colimits (that is, the full subcategory $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant 0} \subseteq \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is closed under filtered colimits).*
3. *The t-structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is Postnikov complete.*
4. *The functor π_0 determines an equivalence of categories from $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})^{\heartsuit} \xrightarrow{\sim} \mathrm{Ab}(\mathcal{X}^{\heartsuit})$.*

Stable homotopy theory on an ∞-topos

Proposition

Let $g^* : \mathcal{X} \to \mathcal{Y}$ be a geometric functor of ∞-topoi (that is, a functor which preserves small colimits and finite limits). Then g^* induces a functor

$$\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \simeq \mathrm{Sp}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{Y}) \simeq \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y}).$$

It is a left adjoint to the pushforward functor $g_* : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$, given by pointwise composition with $g^* : \mathcal{X} \to \mathcal{Y}$.
Since the functor $g^* : \mathcal{X} \to \mathcal{Y}$ preserves n-truncated objects and n-connective objects for every integer n, we conclude that the functor $g^* : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})$ is t-exact: that is, it carries $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geqslant n}$ into $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})_{\geqslant n}$ and $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leqslant n}$ into $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})_{\leqslant n}$. It follows that t-exact.

∞-Connective Sheaves of Spectra

The t-structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is not Whitehead complete in general. For example, there may exist nonzero objects $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ whose all homotopy groups $\pi_n \mathcal{F}$ vanish.

Definition

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ be a sheaf of spectra on $\mathcal{X}$. We will say that $\mathcal{F}$ is ∞-connective if it is n-connective for every integer n. In other words, $\mathcal{F}$ is ∞-connective if $\pi_n \mathcal{F} \simeq 0$ for every integer n.

Remark

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{X}^{\text{hyp}} \subseteq \mathcal{X}$ be the full subcategory spanned by the hypercomplete objects. Then the inclusion map $f_* : \mathcal{X}^{\text{hyp}} \to \mathcal{X}$, which admits a left exact left adjoint $f^* : \mathcal{X} \to \mathcal{X}^{\text{hyp}}$. Hence we obtain a pair of adjoint functors

$$\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \rightleftarrows \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}^{\text{hyp}}).$$

Note that an object $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is ∞-connective if and only if $f^* \mathcal{F} \simeq 0$. Since the faithful, the $f_* : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}^{\text{hyp}}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is also fully faithful.

∞-Connective Sheaves of Spectra

Proposition

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$. The following conditions are equivalent:

1. *The object $\Omega^{\infty}\mathcal{F} \in \mathcal{X}$ is hypercomplete.*
2. *The sheaf of spectra $\mathcal{F}$ belongs to the essential image of the fully faithful embedding $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$*
3. *For every ∞-connective object $\mathcal{G} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$, the mapping space $\mathrm{Map}_{\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})}(\mathcal{G}, \mathcal{F})$ is contractible.*
4. *For every ∞-connective object $\mathcal{G} \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$, every map $u : \mathcal{G} \to \mathcal{F}$ is nullhomotopic.*

Corollary

The left adjoint functor $f^* : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}})$ represents the $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}^{\mathrm{hyp}})$ as the Whitehead completion of the $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$.

Symmetric monoidal structure on sheaves of spectra

For an ∞-topos $\mathcal{X}$, since small colimits commute with pullback in it, it admits a cartesian closed symmetric monoidal structure $\mathcal{X}^{\times}$. Therefore the stabilization $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ admits a natural symmetric monoidal structure $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})^{\otimes}$ (similar to that from $\mathcal{S}$ to Sp).
We now come to sheaves with values in the ∞-category CAlg of $\mathbb{E}_{\infty}$-rings.

Proposition

For any ∞-topos $\mathcal{X}$, we have a canonical equivalence of ∞-categories (even an isomorphism of simplicial sets)

$$\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X}) \simeq \mathrm{CAlg}\left(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})\right).$$

Composing with the forgetful functor $\mathrm{CAlg} \to \mathrm{Sp}$, we obtain a sheaf of spectra on $\mathcal{X}$. In particular, we can define homotopy groups $\pi_n\mathcal{O}$ as previous. These homotopy groups have a bit more structure in this case: $\pi_0\mathcal{O}$ is a commutative ring object in the underlying topos of $\mathcal{X}$, while each $\pi_n\mathcal{O}$ has the structure of a $\pi_0\mathcal{O}$-module.

Sheaves of $\mathbb{E}_\infty$-rings

Definition

We say that a sheaf of $\mathbb{E}_\infty$-rings is connective if it is connective when regarded as a sheaf of spectra on $\mathcal{X}$: that is, if the homotopy groups $\pi_n \mathcal{O}$ vanish for $n < 0$. We let $\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$ spanned by the connective sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Proposition

Let $\mathcal{X}$ be an ∞-topos. Then composition with the truncation functor $\tau_{\geqslant 0} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{cn}}$ induces an equivalence of (symmetric monoidal) ∞-categories $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})^{\mathrm{cn}} \to \mathrm{Shv}_{\mathrm{Sp}^{\mathrm{cn}}}(\mathcal{X})$.

Corollary

Let $\mathcal{X}$ be an ∞-topos. Then composition with the functor $\tau_{\geqslant 0} : \mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{cn}}$ induces an equivalence of ∞-categories $\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})^{\mathrm{cn}} \to \mathrm{Shv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X})$.