The Right Adjunction of Thom spectrum Functor

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0.1 Notation

Let G be a group over the ∞ orthogonal group $O(\infty)$ with $e \in G$ a nondegenerated basepoint, (so G is automatically a group $\mathscr{I}-\mathrm{FCP}$) , $\mathcal{U}/B$ be the (nonbased) CGWH category over a space B , $\mathcal{U}[H]$ be the category of spaces with a right action for a (CGWH) topological group H , $\mathcal{S}$ be the category of May's ($\Omega-$)spectra in a fixed universe. And Thom spectrum functor is denoted by $M : \mathcal{U}/BG \to \mathcal{S}$.

0.2 Proposition

Given a map $(X \to BG) \in \mathcal{U}/BG$ and $E \in \mathcal{S}$, then

$$\mathrm{Hom}_{\mathcal{S}}(MX, E) = \mathrm{Hom}_{\mathcal{U}[G]}(p^*X, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BG}(X, EG \times_G \Omega^\infty E)$$

where p is the map $EG \to BG$. Since $\Omega^\infty E = E_0 = \Omega^V E_V = F(S^V, E_V)$ for any V in the universe, $\Omega^\infty E$ admits a right $G(V)$-action by the left action on S^V, and this action is coherent for all V, so we get a right G-action on $\Omega^\infty E$.

proof:

First we have

$$\mathrm{Hom}_{\mathcal{S}}(MX, E) = \mathrm{Hom}_{\mathcal{S}}(\mathrm{colim}_V MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{S}}(MX_V, E)$$

Second we define EX_V and $Z(V)$ by pullback diagrams,

$$\begin{array}{ccc} EX_V & \longrightarrow & B(*, G(V), G(V)) \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & B(*, G(V), *) \end{array} \qquad \begin{array}{ccc} Z_V & \longrightarrow & B(*, G(V), G) \\ \downarrow & & \downarrow \\ X_V & \longrightarrow & B(*, G(V), *) \end{array}$$

then

$$\lim_V \mathrm{Hom}_{\mathcal{S}}(MX_V, E) = \lim_V \mathrm{Hom}_{\mathcal{U}_*}(EX_{V+} \wedge_{G(V)} S^V, E_V) = \lim_V \mathrm{Hom}_{\mathcal{U}_*[G_{V+}]}(EX_{V+}, \Omega^V E_V)$$

$$= \lim_V \mathrm{Hom}_{\mathcal{U}[G_V]}(EX_V, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[G]}(EX_V \times_{G_V} G, \Omega^\infty E) = \lim_V \mathrm{Hom}_{\mathcal{U}[G]}(Z_V, \Omega^\infty E) = \mathrm{Hom}_G(p^*X, \Omega^\infty E)$$

Since equivariant maps from a principle G-bundle to a G-space is equivalent to following sections.

$$\mathrm{Hom}_G(p^*X, \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/X}(X, p^*X \times_G \Omega^\infty E) = \mathrm{Hom}_{\mathcal{U}/BG}(X, EG \times_G \Omega^\infty E)$$

□

Remark: This process doesn't work in the prespectrum level, because the right G-action on $\Omega^\infty E$ only appears for a spectrum E .