Dualizable Additive Categories

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Start with a familiar example: spectra

Finite spectra

$$\mathrm{Sp} \simeq \mathsf{Ind}(\mathrm{Sp}^{\omega}), \qquad \mathrm{Sp}^{\omega} = \{\text{finite spectra}\}.$$

Every spectrum is a filtered colimit of finite spectra.

Compact objects generate Sp.

Compact generation is a finiteness principle

An object $P \in \mathcal{C}$ is **compact** if

$$\mathrm{Map}_{\mathcal{C}}(P, -) \text{ preserves filtered colimits.}$$

A stable presentable category is **compactly generated** when

$$\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^{\omega}).$$

Homotopy-theoretic examples

For an $\mathbb{E}_1$-ring R, Mod_R is compactly generated.

But sheaf categories and categories in analytic geometry often do not have enough compact objects.

Dualizability: another sort of finiteness

Recall that a spectrum X is dualizable if there is a dual $X^\vee$ with evaluation and coevaluation:

$$X^\vee \otimes X \longrightarrow \mathbb{S}, \qquad \mathbb{S} \longrightarrow X \otimes X^\vee.$$

In spectra:

$$X \text{ dualizable} \iff X \text{ finite}.$$

Key change of viewpoint

Instead of asking whether a *spectrum* is dualizable, ask whether an entire *stable category* is dualizable.

Stable categories form a monoidal category

$$\mathcal{P}\mathrm{r}^{L}_{\mathrm{st}} = \{\text{presentable stable } \infty\text{-categories, left adjoints}\}.$$

It has a tensor product and unit Sp.

$$\mathrm{Fun}^{L}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^{L,L}(\mathcal{C} \times \mathcal{D}, \mathcal{E}).$$

Example

For an $\mathbb{E}_1$-ring R,

$$(\mathrm{Mod}_R)^{\vee} \simeq \mathrm{Mod}_{R^{\mathrm{op}}}.$$

Its evaluation pairing is derived tensor product over R.

This does not mean that every R-module is a dualizable object in Mod_R.

Compact generation gives dualizability

An object $P \in \mathcal{C}$ is **compact** if

$$\mathrm{Map}_{\mathcal{C}}(P, -) \quad \text{preserves filtered colimits.}$$

A stable presentable category is **compactly generated** if

$$\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^{\omega}).$$

Proposition (Lurie)

Every compactly generated presentable stable category is dualizable. In fact,

$$\mathrm{Ind}(\mathcal{A})^{\vee} \simeq \mathrm{Ind}(\mathcal{A}^{\mathrm{op}})$$

for $\mathcal{A}$ small stable, and idempotent-complete.

Beyond compact generation

Compactly assembled: a category $\mathcal{C}$ that is a retract of some compactly generated stable category, through colimit-preserving functors:

$$\mathcal{C} \underset{r}{\overset{i}{\rightleftarrows}} \mathcal{D}, \qquad ri \simeq \mathrm{id}_{\mathcal{C}}.$$

Theorem (Lurie)

$\mathcal{C} \in \mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}$ is dualizable $\iff$ $\mathcal{C}$ is compactly assembled.

A dualizable stable category need not be compactly generated: dualizability is more flexible.

The category of dualizable categories

A colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ is an **internal left adjoint** if its right adjoint F^R also preserves colimits.

$$\mathcal{C} \underset{F^R}{\overset{F}{\rightleftarrows}} \mathcal{D}.$$

Define $\mathrm{Pr}^{\mathrm{dbl}}_{\mathrm{st}}$ using (This is a non-full subcategory of $\mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}$ on dualizable objects.)

objects	dualizable presentable stable categories
morphisms	internal left adjoints

Theorem (Ramzi '24)

$\mathrm{Pr}^{\mathrm{dbl}}_{\mathrm{st}}$ is itself a **presentable** ∞-category.

An example without compact generators

Take the category of sheaves of spectra on the real line:

$$\mathcal{C} = \mathrm{Shv}(\mathbb{R}; \mathrm{Sp}).$$

Surprising facts

$$\mathcal{C} \neq 0, \qquad \mathcal{C}^{\omega} = 0, \qquad \text{but } \mathcal{C} \text{ is dualizable in } \mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}.$$

More generally, sheaf categories on locally compact Hausdorff spaces provide natural examples in Efimov's theory.

Compact generators cannot detect this category, but its *categorical duality* is still meaningful.

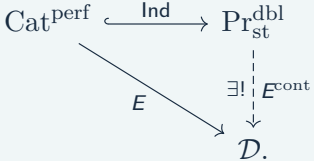
Localizing invariants

A **localizing invariant** $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ is a functor to a stable ∞-category $\mathcal{E}$ sending bifiber (i.e. Karoubi) sequences to fiber sequences:

$$E(\mathcal{A}) \longrightarrow E(\mathcal{B}) \longrightarrow E(\mathcal{C}).$$

Theorem (Efimov '23)

Every localizing invariant $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{D}$ admits a unique continuous extension along Ind:



In particular, the continuous K-theory: $K^{\mathrm{cont}}\big(\mathrm{Ind}(\mathcal{A})\big) \simeq K(\mathcal{A})$.

Efimov: an application to analytic geometry

Let R be a noetherian ring and $I \subset R$ an ideal. The continuous K-theory of the formal completion is

$$K^{\mathrm{cont}}(R, I) := \varprojlim_n K(R/I^n).$$

Theorem (Efimov '25)

$$\boxed{K^{\mathrm{cont}}\left(\mathrm{Nuc}(\widehat{R}_I)\right) \simeq \varprojlim_n K(R/I^n).}$$

Perspective

Continuous K-theory of dualizable categories interacts interestingly with analytic geometry.

Move toward dualizable additive categories

We have seen the chain

compact generated categories $\Longrightarrow$ dualizable stable categories $\Longrightarrow$ continuous K-theory.

But many examples have a natural t-structure:

$$D(R)_{\geq 0} \subset D(R), \qquad \mathrm{Sp}_{\geq 0} \subset \mathrm{Sp}.$$

Question for the main seminar

Can we develop a theory of dualizable categories that also remembers **connectivity**?

The additive world is based on connective spectra

Definition: Additive ∞-categories

Pointed categories with finite biproducts and grouplike mapping spaces:

$$X \oplus Y \simeq X \times Y.$$

The base category is $\mathrm{Sp}_{\geq 0}$, not Sp:

$$\mathcal{P}\mathrm{r}^{L}_{\mathrm{ad}} \simeq \mathrm{Mod}_{\mathrm{Sp}_{\geq 0}}(\mathcal{P}\mathrm{r}^{L}).$$

Connective spectra are the group-like $\mathbb{E}_\infty$-spaces.

A stable/additive dictionary

	Stable	**Additive**
Base	Sp	$\mathrm{Sp}_{\geq 0}$
Presentable categories	$\mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}$	$\mathcal{P}\mathrm{r}^{L}_{\mathrm{ad}}$
Atomic objects	compact	compact projective
Free cocompletion	$\mathsf{Ind}(\mathcal{A})$	$\mathcal{P}_{\Sigma}(\mathcal{A})$

Projective means mapping out preserves geometric realizations; compact projective also preserves filtered colimits.

What is the free additive cocompletion?

If $\mathcal{A}$ is a small additive ∞-category, let

$$\boxed{\mathcal{P}_\Sigma(\mathcal{A}) = \mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}).}$$

Compact projectively generated $\implies$ dualizable additive

Similar to the stable setting,

$$\mathcal{P}_\Sigma(\mathcal{A})$$

is always dualizable additive, whose dual is $\mathcal{P}_\Sigma(\mathcal{A}^{\mathrm{op}})$.

Example: Let R be a connective $\mathbb{E}_1$-ring and fFree_R be the category of finite free R-modules. Then

$$\mathcal{P}_\Sigma(\mathrm{fFree}_R) \simeq \mathrm{Mod}_{R,\geq 0}\,.$$

Definition: dualizable additive category

A presentable additive category $\mathcal{C}$ is **dualizable additive** if it is dualizable in $\mathcal{P}\mathrm{r}^{L}_{\mathrm{ad}}$.

Why this is a connective refinement

Stabilization preserves categorical dualizability:

$$\mathcal{C} \text{ dualizable additive} \quad \Longrightarrow \quad \mathrm{Sp} \otimes \mathcal{C} \text{ dualizable stable.}$$

The converse need not hold: the connective structure imposes extra conditions.

Structure theorem: still a retract

Proposition (Ramzi)

$\mathcal{C}$ is dualizable additive if and only if it compact projectively assembled, i.e. a retract of some $\mathcal{P}_\Sigma(\mathcal{A})$ in $\mathcal{P}\mathrm{r}^L_{\mathrm{ad}}$.

$$\mathcal{C} \xrightarrow{i} \mathcal{P}_\Sigma(\mathcal{A}) \xrightarrow{r} \mathcal{C}, \qquad ri \simeq \mathrm{id}_{\mathcal{C}}.$$

The category need not itself have enough compact projectives.

Connectivity of THH

Stabilization induces a symmetric monoidal functor $\mathrm{Sp}(-) : \mathcal{P}\mathrm{r}^{\mathrm{dbl}}_{\mathrm{ad}} \longrightarrow \mathcal{P}\mathrm{r}^{\mathrm{dbl}}_{\mathrm{st}}$.

Naturality of the categorical trace

Taking traces gives a commutative diagram:

$$\begin{array}{ccc} \mathcal{P}\mathrm{r}^{\mathrm{dbl}}_{\mathrm{ad}} & \xrightarrow{\mathrm{Sp}(-)} & \mathcal{P}\mathrm{r}^{\mathrm{dbl}}_{\mathrm{st}} \\ \Big\downarrow{\scriptstyle \mathrm{tr}} & & \Big\downarrow{\scriptstyle \mathrm{tr}} \\ \mathsf{Sp}_{\geq 0} & \hookrightarrow & \mathrm{Sp}. \end{array}$$

Therefore, for every dualizable additive ∞-category $\mathcal{C}$,

$$\boxed{\mathrm{THH}(\mathrm{Sp}(\mathcal{C})) \simeq \mathrm{tr}(\mathrm{id}_{\mathrm{Sp}(\mathcal{C})}) \in \mathsf{Sp}_{\geq 0}\,.}$$

In particular, its topological Hochschild homology is always connective.

The classical precursor: Grothendieck and Roos

Grothendieck abelian categories

AB5: filtered colimits are exact.

AB4*: arbitrary products are exact.

AB6: products distribute over independently indexed filtered colimits.

Theorem (Clausen–Scholze, Efimov)

A presentable stable category is dualizable stable $\longleftrightarrow$ AB6.

Theorem (Roos; Stefanich–Kanda)

dualizable abelian categories $\longleftrightarrow$ AB5 + AB4* +AB6.

What is a Grothendieck prestable category?

A **prestable** ∞-category behaves like the connective part of a stable category with a t-structure:

$$\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C})_{\geq 0}.$$

Examples: $\mathrm{Sp}_{\geq 0}$ and $\mathrm{Mod}_{R,\geq 0}$ are prestable for a connective ring spectrum R.

Definition

We say $\mathcal{C}$ is separated (hypercomplete) if it has no nonzero infinitely connective objects.

The intrinsic characterization

Theorem (Levy–L.–Sosnilo)

A presentable additive ∞-category $\mathcal{C}$ is dualizable if and only if

1. $\mathcal{C}$ is separated
2. AB5: $\mathcal{C}$ is Grothendieck prestable;
3. AB4*: products of connective objects stay connective in $\mathrm{Sp}(\mathcal{C})$;
4. AB6: filtered colimits distribute with arbitrary products.

Higher almost mathematics

Theorem (Hebestreit–Scholze '24)

Let R be a connective $\mathbb{E}_1$-ring. Connective $\mathbb{E}_1$-ring maps $R \to S$ satisfying

$$S \otimes_R S \xrightarrow{\sim} S, \qquad \pi_0 R \twoheadrightarrow \pi_0 S$$

are classified by idempotent two-sided ideals

$$\boxed{I = I^2 \subseteq \pi_0 R.}$$

The corresponding ring is the *derived quotient* $S \simeq R/I^\infty$. More explicitly, if $J_I = \mathrm{fib}\big(R \to H(\pi_0 R/I)\big)$, then

$$I^\infty := \varprojlim_n J_I^{\otimes R^n}, \qquad R/I^\infty := \mathrm{cofib}(I^\infty \to R).$$

Dualizable additive categories are almost modules

Let R be a connective $\mathbb{E}_1$-ring and $I = I^2 \subseteq \pi_0 R$.

Definition: connective almost modules

$$\boxed{\mathrm{aMod}_{(R,I)}(\mathrm{Sp}_{\geq 0}) := \ker\left(\mathrm{Mod}_{R,\geq 0} \xrightarrow{-\otimes_R R/I^\infty} \mathrm{Mod}_{R/I^\infty,\geq 0}\right).}$$

Theorem (Levy–L.–Sosnilo)

A presentable additive ∞-category $\mathcal{C}$ is dualizable if and only if there exist a connective $\mathbb{E}_1$-ring R and an idempotent two-sided ideal $I \subseteq \pi_0 R$ such that

$$\boxed{\mathcal{C} \simeq \mathrm{aMod}_{(R,I)}(\mathrm{Sp}_{\geq 0}).}$$

Thus every dualizable additive category admits a concrete algebraic realization in almost

Flat objects recover the entire category

Definition

Let $\mathcal{C}$ be a dualizable additive ∞-category. We say an object $x \in \mathcal{C}$ is flat if $F_x^\vee : \mathcal{C}^\vee \to \mathrm{Sp}_{\geq 0}$ is left exact.

We have Lazard theorem:

$$\mathcal{P}_\Sigma(\mathcal{A})^{\mathrm{flat}} \simeq \mathrm{Ind}(\mathcal{A}).$$

Reconstruction theorem (Levy–L.–Sosnilo)

For a dualizable additive category $\mathcal{C}$,

$$\boxed{\mathcal{C} \simeq \mathcal{P}^{\mathrm{small}}_{\sqcup,\mathrm{fil}}(\mathcal{C}^{\mathrm{flat}}).}$$

Here $\mathcal{P}^{\mathrm{small}}_{\sqcup,\mathrm{fil}}$ freely adds colimits while preserving finite sums and filtered colimits already present.

Nuclearity is a trace-class condition

In a compactly generated monoidal stable category, an object M is **nuclear** if

$$P^\vee \otimes M \xrightarrow{\sim} \underline{\mathrm{Hom}}(P, M) \qquad (P \text{ compact}).$$

Definition (Clausen–Scholze)

Let R be a connective adic $\mathbb{E}_\infty$-ring. We define

$$\mathrm{Nuc}(R) := \mathrm{Nuc}(\mathrm{Solid}(R))$$

, where $\mathrm{Solid}(R) := \mathrm{Ind}(\mathcal{D}^b(\mathrm{Proj}_R^{\mathrm{cpl},\omega_1}))$.

Clausen–Scholze's nuclear modules provide the analytic example in our theorem.

The universal property of the category of nuclear modules

Definition: additive rigidity

The tensor unit is compact projective, and the multiplication functor is an internal left adjoint.

Theorem (Ramzi '24)

Let $\mathcal{V} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$. Then the inclusion from $\mathcal{V}$-rigid categories $\mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig}} \hookrightarrow \mathrm{CAlg}_{\mathcal{V}}$ admits a right adjoint $\mathrm{CAlg}_{\mathcal{V}} \xrightarrow{\mathrm{Rig}_{\mathcal{V}}(-)} \mathrm{CAlg}_{\mathcal{V}}^{\mathrm{rig}}$, which called the rigidification.

Theorem (Levy–L.–Sosnilo)

For a connective adic $\mathbb{E}_\infty$-ring R,

$$\boxed{\mathrm{Rig}_{\mathrm{ad}}\left(\mathrm{Mod}_{R,\geq 0}^{\mathrm{cpl}}\right) \simeq \mathrm{Nuc}(R)_{\geq 0}.}$$

Algebraic K-theory as a universal localizing invariant

Theorem (Blumberg–Gepner–Tabuada '13, Efimov '23)

There exists a universal finitary localizing invariant

$$\mathcal{U}_{\mathrm{loc}} : \mathcal{P}\mathrm{r}^{\mathrm{dbl}}_{\mathrm{st}} \longrightarrow \mathcal{M}_{\mathrm{loc}},$$

where $\mathcal{M}_{\mathrm{loc}}$ is a presentable stable ∞-category. More precisely, for every presentable stable $\mathcal{E}$,

$$\mathsf{Fun}^{L}(\mathcal{M}_{\mathrm{loc}}, \mathcal{E}) \simeq \mathsf{Loc}_{\omega}(\mathrm{Cat}^{\mathrm{perf}}, \mathcal{E}).$$

Moreover, its tensor unit corepresents *nonconnective* continuous algebraic K-theory:

$$\boxed{\underline{\mathsf{Map}}_{\mathcal{M}_{\mathrm{loc}}}\big(\mathcal{U}_{\mathrm{loc}}(\mathrm{Sp}), \mathcal{U}_{\mathrm{loc}}(\mathcal{A})\big) \simeq K^{\mathrm{cont}}(\mathcal{A}).}$$

From connective categories to algebraic K-theory

Theorem (Levy–L.–Sosnilo)

There is a universal finitary stable localizing invariant

$$\mathcal{U} : \mathcal{P}\mathrm{r}^{\mathrm{dbl}}_{\mathrm{ad}} \to \mathcal{M}\mathrm{ot}_{\mathrm{pst}},$$

which we call the stable ∞-categoryof prestable motives. Furthermore, its unit corepresents *nonconnective* K-theory:

$$\boxed{\underline{\mathsf{Map}}_{\mathcal{M}\mathrm{ot}_{\mathrm{pst}}}(\mathcal{U}(\mathrm{Sp}_{\geq 0}), \mathcal{U}(\mathcal{A})) \simeq K^{\mathrm{cont}}(\mathrm{Sp}(\mathcal{A})).}$$

The input remembers connectivity, but it represents non-connective K-theory.

Takeaways

1. **Additive dualizability** is a finiteness principle beyond compact projective generation.
2. **Dualizable additive categories** retain features lost under stabilization.
3. **Flat objects** recover dualizable additive categories.
4. **Nuclear modules and motives** connect the theory to analytic geometry and K-theory.

Thank you!